1 (a) Find: $\int \cos x \, dx$

(b) Find: $\int \operatorname{cosec} 3x \cot 3x \, dx$

(c) Find: $\int 2 \sec^2 4x \, dx$

$y = \operatorname{cosec} x$

$\frac{dy}{dx} = -\operatorname{cosec} x \cot x$

$y = \tan x$

$\frac{dy}{dx} = \sec^2 x$

a/ $\sin x + C$

b/ $-\frac{1}{3} \operatorname{cosec} 3x + C$

c/ $\frac{1}{2} \tan 4x + C$

---

2 (a) Express $\tan^2\theta$ in terms of $\sec^2\theta$

(b) Find: $\int \tan^2\theta \, d\theta$

a/ $\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$

$\tan^2\theta + 1 = \sec^2\theta$

$\underline{\tan^2\theta = \sec^2\theta - 1}$

b/ $\int \sec^2\theta - 1 \, d\theta$

$\underline{\tan\theta - \theta + C}$

---

3 (a) Find: $\int \frac{\sin x}{\cos^2 x} \, dx$

(b) Find: $\int 2\operatorname{cosec}^2 x + \sin x \, dx$

a/ $\frac{\sin x}{\cos x} \quad \frac{1}{\cos x}$

$\int \tan x \sec x \, dx = \underline{\sec x + C}$

b/ $\underline{-2\cot x - \cos x + C}$

4 (a) Find: $\displaystyle\int \frac{1}{\cos^2(3x+1)}\,dx$

(b) Find: $\displaystyle\int (\sec x + \tan x)^2\,dx$

a/ $\displaystyle\int \sec^2(3x+1)\,dx$

$\frac{1}{3}\tan(3x+1) + c$

b/ $\displaystyle\int \sec^2 x + 2\sec x\tan x + \tan^2 x\,dx$

$\boxed{1+\tan^2 x = \sec^2 x}$

$\displaystyle\int \sec^2 x + 2\sec x\tan x + \sec^2 x - 1\,dx$

$\displaystyle\int 2\sec^2 x + 2\sec x\tan x - 1\,dx$

$2\tan x + 2\sec x - x + c$

---

5 Find: $\displaystyle\int_0^{\frac{\pi}{4}} \sin 2x\cos 2x\,dx$

$\sin 2x = 2\sin x\cos x$

$\frac{1}{2}\sin 2x = \sin x\cos x$

$\frac{1}{2}\sin 4x = \sin 2x\cos 2x$

$\displaystyle\int_0^{\pi/4} \frac{1}{2}\sin 4x\,dx$

$\left[-\frac{1}{8}\cos 4x\right]_0^{\pi/4}$

$-\frac{1}{8}\cos(\pi) - \left(-\frac{1}{8}\cos(0)\right) = \frac{1}{4}$

6 (a) Express $\sin^2 x$ in terms of $\cos 2x$

(b) Find: $\int \sin^2 x \; dx$

a/

$$\cos 2x = 1 - 2\sin^2 x$$

$$2\sin^2 x = 1 - \cos 2x$$

$$\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

b/

$$\int \frac{1}{2} - \frac{1}{2}\cos 2x \; dx$$

$$\frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

---

7 Find: $\int_0^{\frac{\pi}{2}} \cos^2 x \; dx$

$$\cos 2x = 2\cos^2 x - 1$$

$$1 + \cos 2x = 2\cos^2 x$$

$$\frac{1}{2} + \frac{1}{2}\cos 2x = \cos^2 x$$

$$\int_0^{\pi/2} \frac{1}{2} + \frac{1}{2}\cos 2x \; dx$$

$$\left[\frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$$

$$\left(\frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin \pi\right) - (0) = \frac{1}{4}\pi$$

---

8 Use the substitution $u = x^2 + 4$ to find $\int \frac{2x}{x^2+4} \frac{dx}{du} \, du$

$$\frac{du}{dx} = 2x \qquad \frac{dx}{du} = \frac{1}{2x}$$

$$\int \frac{2x}{u} \frac{1}{2x} \; du$$

$$\int \frac{1}{u} \, du = \ln u + c$$

$$= \ln(x^2+4) + c$$

9 Use the substitution $u = \sin x$ to find $\int \sin^3 x \cos x \frac{dx}{du}\, du$

$$\frac{du}{dx} = \cos x$$

$$\frac{dx}{du} = \frac{1}{\cos x}$$

$$\int \sin^3 x \cos x \cdot \frac{1}{\cos x}\, du$$

$$\int \sin^3 x\, du$$

$$\int u^3\, du$$

$$\frac{1}{4}u^4 + c$$

$$\frac{1}{4}\sin^4 x + c$$

---

10 Use the substitution $u = x^2 + 2$ to find $\int 2x(x^2+2)^2 \frac{dx}{du}\, du$

$$u = x^2 + 2$$

$$\frac{du}{dx} = 2x \qquad \frac{dx}{du} = \frac{1}{2x}$$

$$\int 2x \cdot u^2 \frac{1}{2x}\, du$$

$$\int u^2\, du$$

$$\frac{1}{3}u^3 + c$$

$$\frac{1}{3}(x^2+2)^3 + c$$

11 Use the substitution $u = 1 + e^x$ to find $\int \frac{e^{3x}}{1+e^x} \frac{dx}{du} \, du$

$$\frac{du}{dx} = e^x \qquad \int \frac{e^{3x}}{u} \frac{1}{e^x} \, du$$

$$u - 1 = e^x \qquad \int \frac{e^{2x}}{u} \, du$$

$$(u-1)^2 = e^{2x} \qquad \int \frac{(u-1)^2}{u} \, du$$

$$\int \frac{u^2 - 2u + 1}{u} \, du$$

$$\int u - 2 + \frac{1}{u} \, du$$

$$\frac{1}{2}u^2 - 2u + \ln u + c$$

$$\underline{\underline{\frac{1}{2}(1+e^x)^2 - 2(1+e^x) + \ln(1+e^x) + c}}$$

---

12 Use the substitution $u = x^3 - 4$ to find $\int_2^3 2x^2(x^3-4)^2 \frac{dx}{du} \, du$

$$\frac{du}{dx} = 3x^2 \qquad \text{when } x = 3 \quad u = (3)^3 - 4 = 23$$

$$\frac{dx}{du} = \frac{1}{3x^2} \qquad x = 2 \quad u = (2)^3 - 4 = 4$$

$$\int_4^{23} 2x^2 u^2 \frac{1}{3x^2} \, du$$

$$\int_4^{23} \frac{2}{3} u^2 \, du$$

$$\left[\frac{2}{9}u^3\right]_4^{23}$$

$$\frac{2}{9}(23)^3 - \frac{2}{9}(4)^3 = \underline{\underline{2689.6 \text{ units}^2}}$$

13 Use the substitution $x = \sin u$ to find $\int_0^{0.5} \frac{1}{\sqrt{1-x^2}} \frac{dx}{du} du$

$x = \sin u$

$\frac{dx}{du} = \cos u$

when $x = 0.5$ $\quad 0.5 = \sin u$

$u = \frac{1}{6}\pi$

$x = 0$ $\quad 0 = \sin u$

$u = 0$

$$\int_0^{\frac{1}{6}\pi} \frac{1}{\sqrt{1-\sin^2 u}} \cos u \; du$$

$$\int_0^{\frac{1}{6}\pi} \frac{1}{\cos u} \cos u \; du$$

$$\int_0^{\frac{1}{6}\pi} 1 \; du$$

$$\left[u \right]_0^{\frac{1}{6}\pi}$$

$$\frac{1}{6}\pi - 0$$

$$\underline{\underline{\frac{1}{6}\pi}}$$

14 Use the substitution $u = 1 + \cos x$ to find $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos x} \frac{dx}{du}\, du$

$\frac{du}{dx} = -\sin x$

$u = 1 + \cos \frac{\pi}{2}$

$= 1$

$\frac{dx}{du} = -\frac{1}{\sin x}$

$u = 1 + \cos 0$

$= 2$

$$\int_2^1 \frac{\sin x}{u} \cdot \frac{-1}{\sin x}\, du$$

$$\int_2^1 -\frac{1}{u}\, du$$

$$\int_1^2 \frac{1}{u}\, du$$

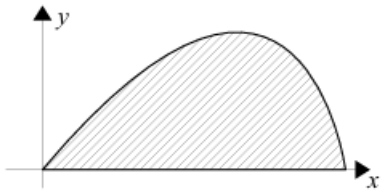
$$\left[\ln u\right]_1^2$$

$$\ln 2 - \ln 1$$

$$\underline{\underline{\ln 2}}$$

15 The curve C has the parametric equations

$$x = 5\cos t \qquad y = 3\sin 2t \qquad 0 \le t \le \frac{\pi}{2}$$



(a) Find the points where the curve meets the x axis

(b) Find the area between the curve and the x axis

a/

$0 = 3\sin 2t$

$0 = \sin 2t$

$2t = 0, \pi$

$t = 0, \frac{\pi}{2}$ $\qquad x = 5\cos 0 = 5$ $\qquad x = 5\cos\frac{\pi}{2} = 0$

$\underline{\underline{(0,0)}}$ and $\underline{\underline{(5,0)}}$

b/

$\int_0^5 y \, dx$ $\qquad x = 5\cos t$, $\quad \frac{dx}{dt} = -5\sin t$

$t=0 \rightarrow 5$, $\quad t=\frac{\pi}{2} \rightarrow 0$

$\int_0^5 3\sin 2t \, \frac{dx}{dt} \, dt$

$\int_{\frac{\pi}{2}}^{0} 3\sin 2t \cdot -5\sin t \, dt$

$\int_{\frac{\pi}{2}}^{0} -15(2\sin t\cos t)\sin t \, dt$

$\int_0^{\frac{\pi}{2}} 30\sin^2 t\cos t \, dt$ $\qquad$ if $y = \sin^3 t$, $\quad \frac{dy}{dx} = 3\sin^2 t\cos t$

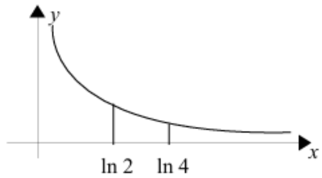
$\left[10\sin^3 t\right]_0^{\frac{\pi}{2}}$

$10\left(\sin\frac{\pi}{2}\right)^3 - 10(\sin 0)^3$

$\underline{\underline{10}}$

16 The curve C has the parametric equations

$$x = \ln(t+2) \qquad y = \frac{1}{t+1} \qquad t > -1$$



The finite region R between by the curve C and the x axis is bounded by the the lines with equations $x = \ln 2$ and $x = \ln 4$

(a) Show that the area of R is given by the integral $\int_0^2 \frac{1}{(t+1)(t+2)}\,dt$

(b) Hence find an exact value for this area

$$\int_{\ln 2}^{\ln 4} y \frac{dx}{dt} \, dt$$

$$\ln 4 = \ln(t+2)$$
$$t = 2$$
$$\ln 2 = \ln(t+2)$$
$$t = 0$$

$$x = \ln(t+2)$$

$$\frac{dx}{dt} = \frac{1}{t+2}$$

$$\int_0^2 \frac{1}{t+1} \cdot \frac{1}{t+2} \, dt$$

$$\int_0^2 \frac{1}{(t+1)(t+2)} \, dt$$

$$\frac{1}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$$

$$1 = A(t+2) + B(t+1)$$

let $t = -2$ $\quad 1 = -B$

$B = 1$

let $t = -1$ $\quad 1 = A$

$$\int_0^2 \frac{1}{t+1} - \frac{1}{t+2} \, dt$$

$$\Big[\ln(t+1) - \ln(t+2) \Big]_0^2$$

$$(\ln 3 - \ln 4) - (\ln 1 - \ln 2)$$

$$\ln 3 - 2\ln 2 + \ln 2 = \ln 3 - \ln 2$$

17 The curve C has the parametric equations

$x = 3\cos 2t \qquad y = 3\sin t \qquad 0 \le t \le \frac{\pi}{4}$

The finite region R between the curve C and the x axis.

(a) The point P is where the curve meets the x axis. Find the coordinates of P.

(b) Show that the area of R is given by the integral $\int_0^{\frac{\pi}{4}} 36\sin^2 t\cos t \; dt$

(c) Hence find an exact value for the area

a/

$0 = 3\sin t$

$0 = \sin t$

$t = 0, \cancel{\pi}$

$x = 3\cos(0)$

$= 3$

b/

$\int_0^3 y \; dx$

$x = 3\cos 2t$

$3 = 3\cos 2t$

$1 = \cos 2t$

$t = 0$

$0 = 3\cos 2t$

$2t = \frac{1}{2}\pi$

$t = \frac{1}{4}\pi$

$\int_{\frac{1}{4}\pi}^{0} 3\sin t \frac{dx}{dt} \; dt$

$\int_{\frac{1}{4}\pi}^{0} 3\sin t \cdot -6\sin 2t \; dt$

$\int_{\frac{1}{4}\pi}^{0} -18\sin t \sin 2t \; dt$

$x = 3\cos 2t$

$\frac{dx}{dt} = -6\sin 2t$

$\int_0^{\frac{1}{4}\pi} 18\sin t \cdot 2\sin t\cos t \; dt$

$\int_0^{\frac{1}{4}\pi} 36\sin^2 t\cos t \; dt$

if $y = \sin^3 t$ $\times 12$

$\frac{dy}{dx} = 3\sin^2 t\cos t$ $\times 12$

$\left[12\sin^3 t\right]_0^{\frac{1}{4}\pi}$

$\left(12\left(\sin\frac{1}{4}\pi\right)^3 - 12(0)\right)$

$3\sqrt{2}$

18 Use integration by parts to find $\int x \sin x \quad dx$

$$\int u \frac{dv}{dx} \, dx = uv - \int v \frac{du}{dx} \, dx$$

$u = x \qquad \frac{dv}{dx} = \sin x$

$\frac{du}{dx} = 1 \qquad v = -\cos x$

$$-x\cos x - \int -\cos x \, dx$$

$$-x \cos x - (-\sin x) + c$$

$$\underline{\underline{\sin x - x\cos x + c}}$$

---

19 Use integration by parts to find $\int 2xe^x \quad dx$

$u = 2x \qquad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = 2 \qquad v = e^x$

$$2xe^x - \int 2e^x \, dx$$

$$\underline{\underline{2xe^x - 2e^x + c}}$$

20 Use integration by parts to find $\int x \sec^2 x \ \mathrm{d}x$

$u = x \qquad \frac{dv}{dx} = \sec^2 x$

$\frac{du}{dx} = 1 \qquad v = \tan x$

$x \tan x - \int \tan x \ dx$

$\underline{x \tan x - \ln|\sec x| + c}$

---

21 Use integration by parts to find $\int x e^{3x} \ \mathrm{d}x$

$u = x \qquad \frac{dv}{dx} = e^{3x}$

$\frac{du}{dx} = 1 \qquad v = \frac{1}{3} e^{3x}$

$\frac{1}{3} x e^{3x} - \int \frac{1}{3} e^{3x} \ dx$

$\underline{\frac{1}{3} x e^{3x} - \frac{1}{9} e^{3x} + c}$

22 Use integration by parts to find the exact value of $\int_0^{\frac{\pi}{6}} 2x\cos x \quad dx$

$u = 2x \qquad \frac{dv}{dx} = \cos x$

$\frac{du}{dx} = 2 \qquad v = \sin x$

$2x\sin x - \int 2\sin x \, dx$

$2x\sin x - (-2\cos x) + C$

$\left[2x \sin x + 2\cos x\right]_0^{\pi/6}$

$\left(2\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}\right)\right) + \left(2\cos\left(\frac{\pi}{6}\right)\right) - \left(0 + 2\cos 0\right)$

$\underline{\underline{\frac{1}{6}\pi + \sqrt{3} - 2}}$

23 Use integration by parts, twice, to find $\int x^2 e^x \, dx$

$u = x^2 \qquad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = 2x \qquad v = e^x$

$x^2 e^x - \int 2x e^x \, dx$

$u = 2x \qquad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = 2 \qquad v = e^x$

$x^2 e^x - \left(2x e^x - \int 2e^x \, dx \right) + c$

$\underline{\underline{x^2 e^x - 2x e^x + 2e^x + c}}$

---

24 Use integration by parts to find $\int \ln x \, dx$

$u = \ln x \qquad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{x} \qquad v = x$

$x \ln x - \int 1 \, dx$

$\underline{\underline{x \ln x - x + c}}$

25 Use integration by parts to find the exact value of $\int_1^2 x^2 \ln x \; dx$

$u = \ln x \qquad \frac{dv}{dx} = x^2$

$\frac{du}{dx} = \frac{1}{x} \qquad v = \frac{1}{3}x^3$

$\frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^2 \, dx$

$\left[\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 \right]_1^2$

$\left(\frac{8}{3}\ln 2 - \frac{8}{9} \right) - \left(0 - \frac{1}{9} \right)$

$\underline{\underline{\frac{8}{3}\ln 2 - \frac{7}{9}}}$

---

26 Integrate with respect to x

(a) $(2x+3)^2$

(b) $\frac{2}{(5x-1)^3}$

a/ $\underline{\underline{\frac{1}{6}(2x+3)^3}}$

b/ $2(5x-1)^{-3}$

$\underline{\underline{-\frac{1}{5}(5x-1)^{-2}}}$

27 Integrate with respect to x

(a) $\sqrt{5x-3}$

(b) e^{2x+3}

a/ $(5x-3)^{\frac{1}{2}}$

$\frac{1}{5} \cdot \frac{2}{3}(5x-3)^{\frac{3}{2}}$

$\frac{2}{15}(5x-3)^{\frac{3}{2}}$

b/ $\frac{1}{2}e^{2x+3}$

---

28 Integrate with respect to x

(a) e^{3-x}

(b) $\dfrac{1}{2x+1}$

a/ $-e^{3-x}$

b/ $\frac{1}{2}\ln(2x+1)$

---

29 Find: $\int_1^2 \dfrac{2}{3x+5}\,dx$

$\left[\frac{2}{3}\ln(3x+5)\right]_1^2$

$\frac{2}{3}\ln 11 - \frac{2}{3}\ln 8$

---

30 Find: $\int_0^1 \frac{10}{(2x+1)^3}\,dx$

$$10(2x+1)^{-3}$$

$$\left[-\frac{5}{2}(2x+1)^{-2}\right]_0^1$$

$$-\frac{5}{2}(3)^{-2} - \left(-\frac{5}{2}(1)^{-2}\right)$$

$$\frac{20}{9}$$

---

31 (a) Express $\frac{3x}{(x+1)(x-2)}$ in partial fractions

(b) Integrate $\frac{3x}{(x+1)(x-2)}$ with respect to x

a/ $$\frac{3x}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$

$$3x = A(x-2) + B(x+1)$$

let $x=2$ $\quad 6 = 3B \quad B=2$

let $x=-1$ $\quad -3 = -3A \quad A=-1$

$$\frac{2}{x-2} - \frac{1}{x+1}$$

b/ $$\int \frac{2}{x-2} - \frac{1}{x+1}\,dx$$

$$2\ln|x-2| - \ln|x+1| + c$$

32 Find the exact value of $\displaystyle\int_0^5 \frac{2x^2 + 3x + 7}{(x+1)^2(x+3)}\,dx$

$$\frac{2x^2 + 3x + 7}{(x+1)^2(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+3)}$$

$$2x^2 + 3x + 7 = A(x+1)(x+3) + B(x+3) + C(x+1)^2$$

let $x = -1$ $\quad 6 = 2B \quad \underline{B = 3}$

let $x = -3$ $\quad 16 = 4C \quad \underline{C = 4}$

let $x = 0$ $\quad 7 = 3A + 3B + C$

$7 = 3A + 9 + 4$

$-6 = 3A \quad \underline{A = -2}$

$$\frac{3}{(x+1)^2} + \frac{4}{x+3} - \frac{2}{x+1}$$

$$\int_0^5 3(x+1)^{-2} + \frac{4}{x+3} - \frac{2}{x+1} \quad dx$$

$$\left[-3(x+1)^{-1} + 4\ln(x+3) - 2\ln(x+1)\right]_0^5$$

$$\left(-3(6)^{-1} + 4\ln 8 - 2\ln 6\right) - \left(-3 + 4\ln 3 - 2\ln 1\right)$$

$$\underline{\underline{\frac{5}{2} + 4\ln 8 - 2\ln 6 - 4\ln 3}}$$

33 The annual rate of increase of a population is equal to 2% of the size of the population. y is the population in millions and t is the time in years.

(a) Write down a differential equation for this relationship

(b) Show that $y = Ae^{0.02t}$ where A is a constant

(c) Given that the initial population is 2.5 million. Find the population after 10 years.

a/ $$\frac{dy}{dt} = 0.02y$$

b/ $$\int \frac{1}{y}\,dy = \int 0.02\,dt$$

$$\ln y = 0.02t + c$$

$$y = e^{0.02t + c}$$

$$y = e^{c} e^{0.02t}$$

$$= \underline{\underline{Ae^{0.02t}}}$$

c/ $A = 2.5$

$$y = 2.5e^{0.02t}$$

when $t = 10$ $$y = 2.5e^{0.02(10)}$$

$$= \underline{3.05 \text{ million}}$$

34 The rate of change of the temperature of a kettle of water (y) after it boils is directly proportional to the difference between the temperature of the water and the room temperature (20°C).

(a) Write down a differential equation for this relationship

(b) Show that $y = 20 + Ae^{kt}$ where A and k are constants

(c) Given that the initial temperature is 100°C write down the value of A.

(d) After 8 minutes the temperature is 60°C show that $k = -\frac{1}{8}\ln 2$

a/
$$\frac{dy}{dt} = k(y - 20)$$

$$\int \frac{1}{y-20}\, dy = \int k \, dt$$

b/
$$\ln(y-20) = kt + c$$

$$y - 20 = e^{kt+c}$$

$$y = 20 + e^{c}e^{kt}$$
$$= 20 + Ae^{kt}$$

c/ $A = 80$ $\quad y = 20 + 80e^{kt}$

d/ $t = 8$ $\quad y = 60$

$$60 = 20 + 80e^{8k}$$
$$40 = 80e^{8k}$$

$$\frac{1}{2} = e^{8k}$$

$$\ln \frac{1}{2} = 8k$$

$$\frac{1}{8}\ln\frac{1}{2} = k$$

$$\frac{1}{8}\ln 2^{-1} = k$$

$$\underline{\underline{-\frac{1}{8}\ln 2 = k}}$$

35 (a) Express $\dfrac{3x-3}{(x+1)(2x-1)}$ in partial fractions

(b) Given that $x > 1$, find the general solution to the differential equation

$$(x+1)(2x-1)\frac{\mathrm{d}y}{\mathrm{d}x} = y(3x-3)$$

(c) Hence find the particular solution to the differential equation that satisfies $y = 6$ at $x = 5$, giving your answer in the form $y = \mathrm{f}(x)$

a/ $$\frac{3x-3}{(x+1)(2x-1)} = \frac{A}{x+1} + \frac{B}{2x-1}$$

$$3x - 3 = A(2x-1) + B(x+1)$$

let $x = -1$ $\quad -6 = -3A$ $\quad$ let $x = \frac{1}{2}$ $\quad -\frac{3}{2} = \frac{3}{2}B$

$A = 2$ $\quad$ $B = -1$

$$\frac{2}{x-1} - \frac{1}{2x-1}$$

b/ $$(x+1)(2x-1)\frac{dy}{dx} = y(3x-3)$$

$$\int \frac{1}{y}\,dy = \int \frac{3x-3}{(x+1)(2x-1)}\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{2}{x-1} - \frac{1}{2x-1}\,dx$$

$$\ln y = 2\ln(x+1) - \frac{1}{2}\ln(2x-1) + c$$

c/ (5,6) $$\ln 6 = 2\ln 6 - \frac{1}{2}\ln 9 + c$$

$$\frac{1}{2}\ln 9 - \ln 6 = c$$

$$\ln 3 - \ln 6 = c$$

$$\ln y = 2\ln(x+1) - \frac{1}{2}\ln(2x-1) + \ln 3 - \ln 6$$

$$\ln y = \ln(x+1)^2 + \ln 3 - \frac{1}{2}\ln(2x-1) - \ln 6$$

$$\ln y = \ln\left(\frac{3(x+1)^2}{6(2x-1)^{\frac{1}{2}}}\right)$$

$$y = \frac{(x+1)^2}{2(2x-1)^{\frac{1}{2}}}$$

36 Find the general solution to the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = xy \sin x$

$$\int \frac{1}{y}\, dy = \int x \sin x \, dx$$

$$u = x \qquad \frac{dv}{dx} = \sin x$$

$$\frac{du}{dx} = 1 \qquad v = -\cos x$$

$$\ln y = -x\cos x - \int -\cos x \, dx$$

$$\ln y = -x\cos x + \sin x + c$$

37 Find the general solution to the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 \ln x$

$$\int \frac{1}{y^2}\, dy = \int \ln x \, dx$$

$$u = \ln x \qquad \frac{dv}{dx} = 1$$

$$\frac{du}{dx} = \frac{1}{x} \qquad v = x$$

$$-y^{-1} = x \ln x - \int 1 \, dx$$

$$-\frac{1}{y} = x \ln x - x + c$$

38 Find the solution to the differential equation $\dfrac{dy}{dx} = (y+1)^2$ given that when $y = 0$, $x = 2$

Give your answer in the form $y = f(x)$

$$\int \frac{1}{(y+1)^2}\, dy = \int 1 \, dx$$

$$\int (y+1)^{-2}\, dy = \int 1 \, dx$$

$$-(y+1)^{-1} = x + c$$

at $(2, 0)$ $$-1 = 2 + c$$

$$c = -3$$

$$-\frac{1}{y+1} = x - 3$$

$$-(y+1) = \frac{1}{x-3}$$

$$y + 1 = -\frac{1}{x-3}$$

$$\underline{\underline{y = -\frac{1}{x-3} - 1}}$$

or... $$y = \frac{1}{3-x} - \frac{(3-x)}{3-x}$$

$$= \underline{\underline{\frac{x-2}{3-x}}}$$

39 The height of a Ferris wheel above the ground, H metres is modelled by the differential equation:

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{H\sin(0.2t)}{5}$$

where t is the time, in minutes, from the start of the ride.

Given that the passenger is $\frac{48}{e}$ m above the ground at the start of the ride,

(a) Show that $H = 48e^{-\cos(0.2t)}$ **(5)**

(b) Find the time, to the nearest minute, it takes for the ride to reach its maximum height. **(2)**

a/ $$\int \frac{1}{H}\, dH = \int \frac{1}{5}\sin(0.2t)\, dt$$

$$\ln H = -\cos(0.2t) + c$$

$t=0 \quad H = \frac{48}{e} \qquad \ln\frac{48}{e} = -1 + c$

$$\ln(48) - 1 = -1 + c$$
$$c = \ln 48$$

$$\ln H = -\cos(0.2t) + \ln 48$$

$$H = e^{-\cos 0.2t + \ln 48}$$

$$H = e^{\ln 48} e^{-\cos 0.2t}$$

$$\underline{\underline{H = 48e^{-\cos 0.2t}}}$$

b/ max height when $-\cos 0.2t = 1$

$$\cos 0.2t = -1$$
$$0.2t = \pi$$
$$t = 5\pi$$
$$= \underline{\underline{16 \text{ minutes}}}$$

40 Show that $\int_0^3 \frac{5x}{\sqrt{x+1}}\,dx = \frac{40}{3}$

$$\int_0^3 5x(x+1)^{-\frac{1}{2}}\,dx$$

$$u = 5x \qquad \frac{dv}{dx} = (x+1)^{-\frac{1}{2}}$$

$$\frac{du}{dx} = 5 \qquad v = 2(x+1)^{\frac{1}{2}}$$

$$10x(x+1)^{\frac{1}{2}} - \int 10(x+1)^{\frac{1}{2}}\,dx$$

$$\left[10x(x+1)^{\frac{1}{2}} - \frac{20}{3}(x+1)^{\frac{3}{2}}\right]_0^3$$

$$\left(10(3)(4)^{\frac{1}{2}} - \frac{20}{3}(4)^{\frac{3}{2}}\right) - \left(0 - \frac{20}{3}(1)\right)$$

$$\left(60 - \frac{160}{3}\right) + \frac{20}{3}$$

$$60 - \frac{140}{3}$$

$$\frac{180}{3} - \frac{140}{3}$$

$$\frac{40}{3}$$

41 The rate of decrease in temperature of water in a kettle t minutes after it switches off is given by:

$$\frac{d\theta}{dt} = -k(\theta - 20)$$

Where θ is the temperature of the water in degrees Celsius and k is a constant to be found.

Given that when t = 0, $\theta = 98$ and when t = 5, $\theta = 59$

Find a model for θ in the form $\theta = Ae^{kt} + B$, stating the values of A, B and k.

$$\int \frac{1}{\theta - 20}\, d\theta = \int -k \, dt$$

$$\ln(\theta - 20) = -kt + c$$

when $t = 0$ $\theta = 98$ $\therefore$ $\ln 78 = c$

$$\ln(\theta - 20) = -kt + \ln 78$$

when $t = 5$ $\theta = 59$ $\ln 39 = -5k + \ln 78$

$$\ln 39 - \ln 78 = -5k$$

$$\ln \frac{1}{2} = -5k$$

$$k = -\frac{1}{5} \ln \frac{1}{2}$$

$$= \frac{1}{5} \ln 2$$

$$\theta = Ae^{kt} + B$$

$A = 78$ $B = 20$ $k = \frac{1}{5} \ln 2$

42 The rate of increase of a population, P, t days after it is initially recorded is given by:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{P}{5}$$

Given that the initial population was 50.

Find a model for P in the form $P = Ae^{kt}$

$$\int \frac{1}{P}\,dP = \int \frac{1}{5}\,dt$$

$$\ln P = \frac{1}{5}t + c$$

when $t = 0$ $P = 50$

$$\ln 50 = c$$

$$\ln P = \frac{1}{5}t + \ln 50$$

$$P = e^{\frac{1}{5}t + \ln 50}$$

$$P = e^{\ln 50} \cdot e^{\frac{1}{5}t}$$

$$\underline{\underline{P = 50e^{\frac{1}{5}t}}}$$

43 After it rains a puddle has 40 cm³ of water.

Two hours later the puddle has 30 cm³ of water.

In a simple model the rate of decrease of the volume of water is inversely proportional to the square of the volume of water in the puddle.

(a) Find an equation linking the volume of water in the puddle and time. **(5)**

(b) Hence find the total time taken for the water in the puddle to fully evaporate. Give your answer in hours and minutes, to the nearest minute. **(3)**

(c) Suggest a limitation of the model. **(1)**

$$\frac{dv}{dt} = -\frac{k}{v^2}$$

$$\int v^2 \, dv = \int -k \, dt$$

$$\frac{1}{3}v^3 = -kt + c$$

when $t=0$ $v=40$

$$\frac{1}{3}(40)^3 = c$$

$$c = \frac{64000}{3}$$

$$\frac{1}{3}v^3 = -kt + \frac{64000}{3}$$

when $t=2$ $v=30$

$$\frac{1}{3}(30)^3 = -2k + \frac{64000}{3}$$

$$9000 = -2k + \frac{64000}{3}$$

$$2k = \frac{37000}{3}$$

$$k = \frac{18500}{3}$$

$$\frac{1}{3}v^3 = -\frac{18500}{3}t + \frac{64000}{3}$$

$$v^3 = -18500t + 64000$$

b/ $v = 0$

$$0 = -18500t + 64000$$

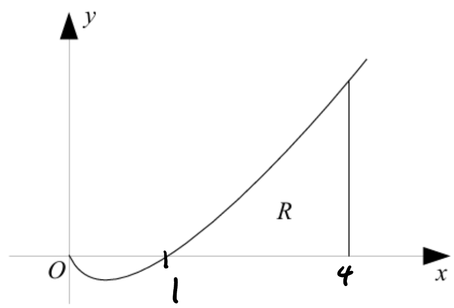
$$18500t = 64000$$

$$t = 3.46 \text{ hours}$$

$$= \underline{3 \text{ hours } 28 \text{ mins}}$$

c/ The changing weather would affect the volume of water. If sunny it would evaporate faster, if it rains it would get bigger.

44



The diagram shows the curve $y = x \ln x, x > 0$

The region R is bounded by the curve, the line $x = 4$ and the x-axis.

Show that the exact area of $R = A\ln 2 + B$ where A and B are rational numbers to be found.

crosses x when $y=0$ $\quad 0 = x \ln x$

$x = 0 \quad \ln x = 0$

$x = 1$

$$\int_1^4 x \ln x \, dx$$

$u = \ln x \qquad \frac{dv}{dx} = x$

$\frac{du}{dx} = \frac{1}{x} \qquad v = \frac{1}{2}x^2$

$$\frac{1}{2}x^2 \ln x - \int \frac{1}{2}x \, dx$$

$$\left[\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right]_1^4$$

$$\left(\frac{1}{2}(16)\ln 4 - \frac{1}{4}(16)\right) - \left(\frac{1}{2}(1)\ln 1 - \frac{1}{4}\right)$$

$$8 \ln 4 - 4 + \frac{1}{4}$$

$$8 \ln 2^2 - \frac{15}{4}$$

$$\underline{\underline{16 \ln 2 - \frac{15}{4}}}$$

45 $y = (x^2 - 3)^5$

(a) Find $\frac{dy}{dx}$

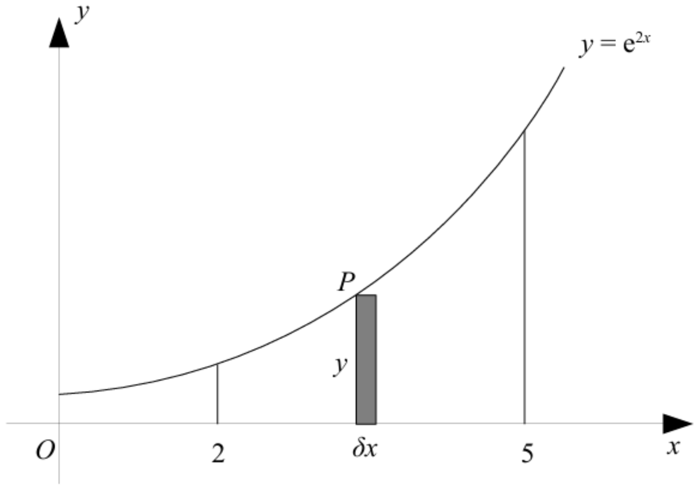
(b) Hence find $\int 2x(x^2 - 3)^4 \, dx$

a/ $\frac{dy}{dx} = 5(x^2 - 3)^4 \times 2x$

$= \underline{\underline{10x(x^2 - 3)^4}}$

b/ $\underline{\underline{\frac{1}{5}(x^2 - 3)^5 + c}}$

46



The diagram shows the curve $y = e^{2x}$, $x > 0$

The point $P(x, y)$ lies on the curve.
The rectangle, shown shaded on the diagram, has height y and width δx

Calculate

$$\lim_{\delta x \to 0} \sum_{x=2}^{5} e^{2x} \delta x$$

$$\int_2^5 e^{2x} \, dx$$

$$\left[\frac{1}{2} e^{2x} \right]_2^5$$

$$\underline{\underline{\frac{1}{2}e^{10} - \frac{1}{2}e^4}}$$

47 Use the substitution $u = 2 - \sqrt{x}$ to show that

$$\int \frac{1}{2-\sqrt{x}} = -4\ln|2-\sqrt{x}| - 2\sqrt{x} + k$$

where k is a constant.

$$u = 2 - x^{\frac{1}{2}}$$

$$\frac{du}{dx} = -\frac{1}{2}x^{-\frac{1}{2}}$$

$$= -\frac{1}{2\sqrt{x}}$$

$$\frac{dx}{du} = -2\sqrt{x}$$

$$\sqrt{x} = 2-u$$

$$\int \frac{1}{2-\sqrt{x}} \frac{dx}{du}\, du$$

$$\int \frac{1}{u} \cdot -2\sqrt{x}\, du$$

$$\int \frac{-2(2-u)}{u}\, du$$

$$\int \frac{-4}{u} + \frac{2u}{u}$$

$$\int \frac{-4}{u} + 2\, du$$

$$-4\ln u + 2u + c$$

$$-4\ln|2-\sqrt{x}| + 2(2-\sqrt{x}) + c$$

$$-4\ln|2-\sqrt{x}| + 4 - 2\sqrt{x} + c$$

let $4+c=k$

$$\underline{\underline{-4\ln|2-\sqrt{x}| - 2\sqrt{x} + k}}$$

48 (a) Use the substitution $x = u^2 + 1$ to show that

$$\int_2^{10} \frac{2\,\mathrm{d}x}{(x-1)(2+\sqrt{x-1})} = \int_q^p \frac{4\,\mathrm{d}u}{u(2+u)}$$

where p and q are positive constants to be found.

(b) Hence, using algebraic integration, show that

$$\int_2^{10} \frac{2\,\mathrm{d}x}{(x-1)(2+\sqrt{x-1})} = \ln a$$

where a is a rational constant to be found.

$10 = u^2 + 1$
$u = 3$

$$\int_2^{10} \frac{2}{(x-1)(2+\sqrt{x-1})} \frac{dx}{du} \, du$$

$2 = u^2 + 1$
$u = 1$

$$\frac{dx}{du} = 2u$$

$$\int_1^3 \frac{2}{u^2(2+u)} \cdot 2u \, du$$

$$\int_1^3 \frac{4u}{u^2(2+u)} \, du$$

$$\int_1^3 \frac{4\,du}{u(u+2)}$$

b/

$$\frac{4}{u(u+2)} = \frac{A}{u} + \frac{B}{u+2}$$

$$4 = A(u+2) + Bu$$

let $u = 0$ $\quad 4 = 2A \quad A = 2$

let $u = -2$ $\quad 4 = -2B \quad B = -2$

$$\int_1^3 \frac{2}{u} - \frac{2}{u+2} \, du$$

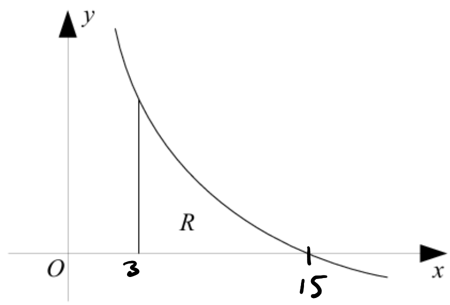
$$\Big[2\ln u - 2\ln(u+2) \Big]_1^3$$

$$2\ln 3 - 2\ln 5 - (2\ln 1 - 2\ln 3)$$

$$4\ln 3 - 2\ln 5$$

$$\ln 81 - \ln 25 = \ln\left(\frac{81}{25}\right)$$

49



The curve C with equation $y = \dfrac{15 - x}{(2x-3)(x+1)}$

The region R is bounded by the curve C, the x-axis and the line with equation $x = 3$

Show that the exact value of the area of R is $a \ln 2 + b \ln 3$, where a and b are rational constants to be found.

crosses x when $y=0$ $\quad 0 = \dfrac{15 - x}{(2x-3)(x+}$

$0 = 15 - x$

$x = 15$

$$\int_3^{15} \frac{15 - x}{(2x-3)(x+1)}\, dx$$

$$\frac{15-x}{(2x-3)(x+1)} = \frac{A}{2x-3} + \frac{B}{x+1}$$

$$15 - x = A(x+1) + B(2x-3)$$

let $x = -1$ $\quad 16 = -5B$

$B = -\frac{16}{5}$

let $x = \frac{3}{2}$ $\quad 13.5 = 2.5A$

$A = \frac{27}{5}$

$$\frac{1}{5}\int_3^{15} \frac{27}{2x-3} - \frac{16}{x+1}\, dx$$

$$\frac{1}{5}\left[\frac{27}{2}\ln(2x-3) - 16\ln(x+1)\right]_3^{15}$$

$$\left[\frac{27}{10}\ln(2x-3)-\frac{16}{5}\ln(x+1)\right]_3^{15}$$

$$\left(\frac{27}{10}\ln 27-\frac{16}{5}\ln 16\right)-\left(\frac{27}{10}\ln 3-\frac{16}{5}\ln 4\right)$$

$$\left(\frac{27}{10}\ln 3^3-\frac{16}{5}\ln 2^4-\frac{27}{10}\ln 3+\frac{16}{5}\ln 2^2\right)$$

$$\frac{81}{10}\ln 3-\frac{64}{5}\ln 2-\frac{27}{10}\ln 3+\frac{32}{5}\ln 2$$

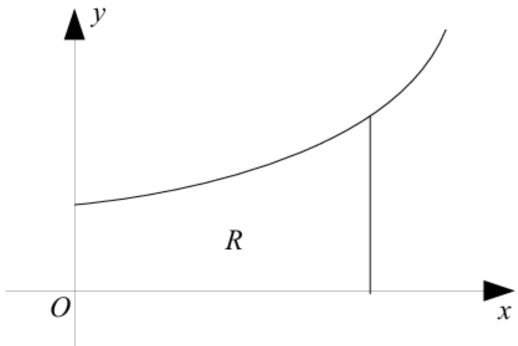
$$\underline{\underline{\frac{54}{10}\ln 3-\frac{32}{5}\ln 2}}$$

$$a=-\frac{32}{5}\qquad b=\frac{54}{10}$$

50 (a) Complete the table below, giving values of $\sqrt{2^x + 1}$ to 3 decimal places.

x	0	0.5	1	1.5	2	2.5	3
$\sqrt{2^x + 1}$	1.414	1.554	1.732	1.957	2.236	2.580	3

The region R which is bounded by the curve with equation $y = \sqrt{2^x + 1}$, the x-axis and the lines $x = 0$ and $x = 3$



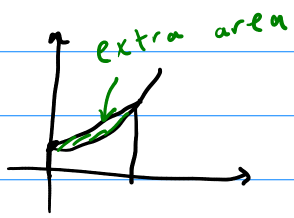
(b) Use the trapezium rule to find an approximation for the area of R.

(c) State, giving a reason, whether your approximation in part (b) is an overestimate or an underestimate for the area of R.

b/ $0.5\left(\frac{1.414}{2} + 1.554 + 1.732 + 1.957 + 2.236 + 2.580 + \frac{3}{2}\right)$

$= 6.133 \text{ units}^2$

c/ overestimate as the curve is convex. The trapeziums will go above the curve.



51 A spherical balloon is being inflated.

At time t seconds the balloon has a radius r cm and a volume V cm^3.

The volume of the balloon is modelled as increasing at a constant rate.

(a) Show that
$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{k}{r^2}$$
where k is a positive constant.

Given that the balloon is initially empty and after 5 seconds the radius of the balloon is 4 cm.

(b) Solve the differential equation to find an equation linking r and t.

(c) Suggest a limitation of the model.

a/ $\frac{dv}{dt} = c$ $\quad v = \frac{4}{3}\pi r^3$

$\frac{dv}{dr} = 4\pi r^2$

$\frac{dr}{dt} = \frac{dr}{dv} \times \frac{dv}{dt}$

$= \frac{1}{4\pi r^2} \times c$ $\quad$ let $\frac{c}{4\pi} = k$

$= \frac{k}{r^2}$

b/ $\frac{dr}{dt} = \frac{k}{r^2}$

$\int r^2 \, dr = \int k \, dt$

$\frac{1}{3}r^3 = kt + c$ $\quad t = 0 \ r = 0 \quad \therefore \ c = 0$

$\frac{1}{3}r^3 = kt$

$t = 5, \ r = 4 \quad \frac{1}{3}(4)^3 = 5k$

$k = \frac{64}{15}$

$\frac{1}{3}r^3 = \frac{64}{15}t$

$r^3 = \frac{64}{5}t$

c/ The balloon cannot keep inflating. It will pop.

52 The table below shows corresponding values of x and y for $y = \sqrt{\frac{2+x}{1+x}}$

The values of y are given to 4 significant figures.

x	0	0.5	1	1.5	2
y	1.414	1.291	1.225	1.183	1.155

(a) Use the trapezium rule, with all the values of y in the table, to find an estimate for

$$\int_0^2 \sqrt{\frac{2+x}{1+x}}\,dx$$

giving your answer to 3 significant figures.

(b) Using your answer to part (a), deduce an estimate for $\int_0^2 \sqrt{\frac{8+4x}{1+x}}\,dx$

a/ $0.5\left(\frac{1.414}{2} + 1.291 + 1.225 + 1.183 + \frac{1.155}{2}\right)$

$= \underline{\underline{2.49}}$

b/ $\sqrt{4}\sqrt{\frac{2+x}{1+x}}$

$2\sqrt{\frac{2+x}{1+x}}$

$2 \times 2.49 = \underline{\underline{4.98}}$

53 (a) Given that $\dfrac{2x^2 - x + 1}{x + 1} \equiv Ax + B + \dfrac{C}{x + 1}$ $\quad x \neq -1$

find the values of the constants A, B and C.

(b) Hence, using algebraic integration, find the exact value of

$$\int_0^7 \frac{2x^2 - x + 1}{x + 1}\, dx$$

giving your answer in the form $a + b \ln 2$ where a and b are integers to be found.

a)

$$\begin{array}{r|l} & 2x - 3 \\ \hline x+1 & 2x^2 - x + 1 \\ & 2x^2 + 2x \\ & \quad -3x + 1 \\ & \quad -3x - 3 \\ \hline & \qquad\quad 4 \end{array}$$

$$2x - 3 + \frac{4}{x+1}$$

$$\int_0^7 2x - 3 + \frac{4}{x+1}\, dx$$

$$\Big[x^2 - 3x + 4\ln(x+1)\Big]_0^7$$

$$(7^2 - 3(7) + 4\ln 8) - (0)$$

$$28 - 4\ln 8$$

$$28 - 4\ln 2^3$$

$$\underline{\underline{28 - 12\ln 2}}$$

54 A curve C has equation $y = \mathrm{f}(x)$

Given that

- $\mathrm{f}'(x) = 6x^2 + ax + 7$ where a is a constant
- the y intercept of C is -20
- $(x + 4)$ is a factor of $\mathrm{f}(x)$

find, in simplest form, $\mathrm{f}(x)$

$$f(x) = \frac{6x^3}{3} + \frac{ax^2}{2} + 7x + c$$

$$= 2x^3 + \frac{ax^2}{2} + 7x + c$$

$c = -20$ $\therefore$ $f(x) = 2x^3 + \frac{ax^2}{2} + 7x - 20$

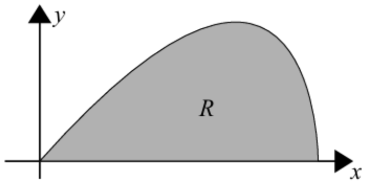
$f(-4) = 0$ $8a - 176 = 0$

$\underline{\underline{a = 22}}$

$$\underline{\underline{f(x) = 2x^3 + 11x^2 + 7x - 20}}$$

55 The curve C has the parametric equations

$$x = 2\sin t \qquad y = 3\sin 2t \qquad 0 \le t \le \frac{\pi}{2}$$



(a) Show that the area of R is given by $\int_0^{\frac{\pi}{2}} 12\sin t\cos^2 t \; dt$

(b) Hence show, by algebraic integration, that the area of R is exactly 4

a/ crosses x when $y=0$

$0 = 3\sin 2t$

$\sin 2t = 0$

$2t = 0, \pi$

$t = \underline{\underline{0}}, \underline{\underline{\frac{\pi}{2}}}$

$\int_0^{\frac{\pi}{2}} y \frac{dx}{dt} \, dt$

$x = 2\sin t$

$\frac{dx}{dt} = 2\cos t$

$\int_0^{\frac{\pi}{2}} 3\sin 2t \cdot 2\cos t \, dt$

$\int_0^{\frac{\pi}{2}} 6\sin 2t\cos t \, dt$

$\sin 2t = 2\sin t\cos t$

$\underline{\underline{\int_0^{\frac{\pi}{2}} 12\sin t\cos^2 t \, dt}}$

b/ if $y = \cos^3 t$

$\times -4$

$\frac{dy}{dt} = 3\cos^2 t\,(-\sin t)$

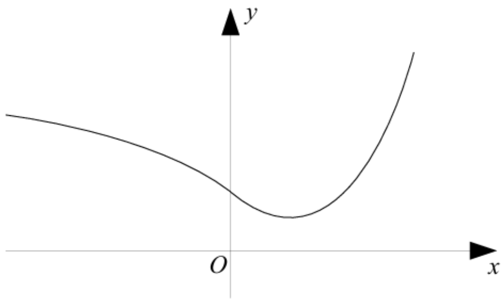
$= -3\sin t\cos^2 t$

$\times -4$

$\left[-4\cos^3 t\right]_0^{\frac{\pi}{2}}$

$-4\left(\cos\frac{\pi}{2}\right)^3 - \left(-4(\cos(0))^3\right) = \underline{\underline{4}}$

56 The curve C with equation $y = \mathrm{f}(x)$ is shown on the diagram.



Given the minimum point of C is $(1, \mathrm{e})$ and $\mathrm{f}'(x) = 2(x - 1)\mathrm{e}^x$ find an equation for $\mathrm{f}(x)$.

$u = 2x - 2$ $\qquad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = 2$ $\qquad v = e^x$

$$f(x) = e^x(2x-2) - \int 2e^x \, dx$$

$$= e^x(2x-2) - 2e^x + c$$

$$= 2e^x x - 2e^x - 2e^x + c$$

$$= 2e^x x - 4e^x + c$$

$(1, e)$ $\qquad e = 2e - 4e + c$

$c = 3e$

$$\underline{\underline{f(x) = 2xe^x - 4e^x + 3e}}$$

57 (a) Prove the identity $\dfrac{\sin 2x}{1+\cot^2 x} \equiv 2\sin^3 x \cos x$

(b) Hence, find $\displaystyle\int \frac{4\sin 4\theta}{1+\cot^2 2\theta}\, d\theta$

$\sin 2x = 2\sin x \cos x$ $\qquad$ $1+\cot^2 x = \operatorname{cosec}^2 x$

$$\frac{2\sin x \cos x}{\operatorname{cosec}^2 x}$$

$$2\sin x \cos x \cdot \sin^2 x$$

$$\underline{\underline{2\sin^3 x \cos x}}$$

b) $\displaystyle\int 2\sin^3 2\theta \cos 2\theta \; d\theta$ $\qquad$ if $y = \sin^4 2\theta$ $\times \frac{1}{4}$

$$\underline{\underline{\frac{1}{4}\sin^4 2\theta}}$$

$\dfrac{dy}{dx} = 4\sin^3 2\theta \times 2\cos 2\theta$

$= 8\sin^3 2\theta \cos 2\theta \times \frac{1}{4}$

58 Show that $\int_0^6 \frac{4x^2}{x+2}\, dx = A + B \ln 2$

where A and B are integers to be found.

$$\frac{4x^2}{x+2} = Ax + B + \frac{C}{x+2}$$

$$4x^2 = Ax(x+2) + B(x+2) + C$$

let $x = -2$ $\quad 16 = C$

let $x = 0$ $\quad 0 = 2B + 16$

$B = -8$

let $x = 1$ $\quad 4 = 3A + 3(-8) + 16$

$A = 4$

$$\int_0^6 4x - 8 + \frac{16}{x+2}\, dx$$

$$\left[2x^2 - 8x + 16 \ln(x+2) \right]_0^6$$

$$(24 + 16 \ln 8) - (16 \ln 2)$$

$$24 + 16 \ln 2^3 - 16 \ln 2$$

$$24 + 48 \ln 2 - 16 \ln 2$$

$$\underline{\underline{24 + 32 \ln 2}}$$

59 (a) $y = e^x (\sin x - \cos x)$

Find $\dfrac{dy}{dx}$

Simplify your answer.

(b) Hence, find $\int e^x \sin x \; dx$

a/ $u = e^x \qquad v = \sin x - \cos x$

$\dfrac{du}{dx} = e^x \qquad \dfrac{dv}{dx} = \cos x + \sin x$

$\dfrac{dy}{dx} = e^x(\cos x + \sin x) + e^x(\sin x - \cos x)$

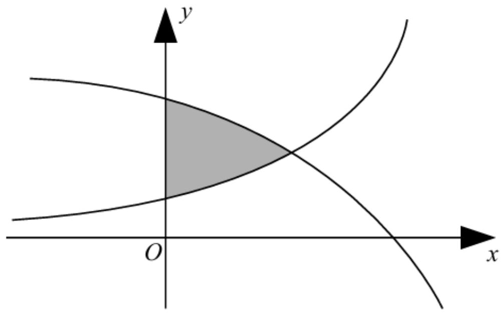
$= e^x(\cos x + \sin x + \sin x - \cos x)$

$= e^x(2\sin x)$

$= 2\sin x \, e^x$

b/ $\dfrac{1}{2}e^x(\sin x - \cos x)$

60 The region enclosed by the y-axis, $y = e^{2x}$ and $y = 12 - e^x$ is shown on the diagram.



Find the exact area of the shaded region.

intersection where $e^{2x} = 12 - e^x$

$$e^{2x} + e^x - 12 = 0$$

$$(e^x + 4)(e^x - 3)$$

$$e^x = -4 \quad e^x = 3$$

$$\times$$

$$x = \ln 3$$

$$\int_0^{\ln 3} 12 - e^x - e^{2x} \, dx$$

$$\left[12x - e^x - \tfrac{1}{2} e^{2x} \right]_0^{\ln 3}$$

$$\left(12\ln 3 - e^{\ln 3} - \tfrac{1}{2} e^{2\ln 3} \right) - \left(12(0) - e^0 - \tfrac{1}{2} e^0 \right)$$

$$12 \ln 3 - 3 - \tfrac{1}{2} e^{\ln 9} + 1 + \tfrac{1}{2}$$

$$12 \ln 3 - 3 - \tfrac{9}{2} + 1 + \tfrac{1}{2}$$

$$\underline{12 \ln 3 - 6}$$

61 Use integration by substitution to show that $\int_1^2 x\sqrt{5x-1}\,dx = \frac{1456}{375}$

$u = 5x - 1$ when $x = 2$ $u = 9$

$\frac{du}{dx} = 5$ $x = 1$ $u = 4$

$\frac{dx}{du} = \frac{1}{5}$

$$\int_4^9 x\sqrt{u} \cdot \frac{1}{5}\,du$$

$x = \frac{u+1}{5}$

$$\int_4^9 \frac{u+1}{5} \cdot u^{\frac{1}{2}} \cdot \frac{1}{5}\,du$$

$$\int_4^9 \frac{u^{\frac{3}{2}} + u^{\frac{1}{2}}}{25}\,du$$

$$\frac{1}{25}\int_4^9 u^{3/2} + u^{\frac{1}{2}}\,du$$

$$\frac{1}{25}\left[\frac{2}{5}u^{5/2} + \frac{2}{3}u^{\frac{3}{2}}\right]_4^9$$

$$\frac{1}{25}\left[\left(\frac{2}{5}(9)^{\frac{5}{2}} + \frac{2}{3}(9)^{\frac{3}{2}}\right) - \left(\frac{2}{5}(4)^{\frac{5}{2}} + \frac{2}{3}(4)^{\frac{3}{2}}\right)\right]$$

$$\frac{1}{25}\left(\frac{576}{5} - \frac{272}{15}\right)$$

$$\underline{\underline{\frac{1456}{375}}}$$

62 (a) Use the trapezium rule, with four strips each of width 0.5, to find an estimate for

$$\int_1^3 x\sqrt{e^x}\, dx$$

(b) Explain how the trapezium rule could be used to obtain a more accurate estimate.

x	1	1.5	2	2.5	3
y	1.649	3.176	5.437	8.726	13.445

$$0.5\left(\frac{1.649}{2} + 3.176 + 5.437 + 8.726 + \frac{13.445}{2}\right)$$

12.4 units^2

b/ More strips could be used.

---

63 Show that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{4\sin 2x}{3 - \cos(2x)}\, dx = \ln 4$

if $y = \ln(3 - \cos 2x)$ $\times 2$

$$\frac{dy}{dx} = \frac{1}{3-\cos 2x} \cdot 2\sin 2x$$

$$= \frac{2\sin 2x}{3 - \cos 2x} \quad \times 2$$

$$\Big[2\ln(3 - \cos 2x)\Big]_0^{\frac{\pi}{2}}$$

$2\ln 4 - 2\ln 2$

$2\ln 2^2 - 2\ln 2$

$4\ln 2 - 2\ln 2$

$\underline{\underline{2\ln 2}}$

64 Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\ln x}{2x^2 y} \qquad \text{for } x > 0$$

Given $x = 1$ when $y = 1$

Write your answer in the form $y^2 = \mathrm{f}(x)$

$$\frac{dy}{dx} = \frac{\ln x}{2x^2 y}$$

$$\int 2y \, dy = \int x^{-2} \ln x \, dx$$

$$u = \ln x \qquad \frac{dv}{dx} = x^{-2}$$

$$\frac{du}{dx} = \frac{1}{x} \qquad v = -x^{-1}$$

$$y^2 = -\frac{1}{x} \ln x - \int -x^{-2} \, dx$$

$$y^2 = -\frac{1}{x} \ln x - x^{-1} + c$$

$$y^2 = -\frac{1}{x} \ln x - \frac{1}{x} + c \qquad (1,1)$$

$$1 = -1 + c$$

$$c = 2$$

$$y^2 = -\frac{1}{x} \ln x - \frac{1}{x} + 2$$

65 (a) Express $\dfrac{4x+5}{(x+2)^2}$ in the form $\dfrac{A}{(x+2)} + \dfrac{B}{(x+2)^2}$

(b) Show that $\displaystyle\int_1^4 \frac{4x+5}{(x+2)^2}\,dx = a + b\ln 2$

where a and b are rational numbers.

$$4x+5 = A(x+2) + B$$

let $x=-2$ $\quad -3 = B$

let $x=0$ $\quad 5 = 2A - 3$

$A = 4$

$$\frac{4}{x+2} - \frac{3}{(x+2)^2}$$

b/

$$\int_1^4 \frac{4}{x+2} - 3(x+2)^{-2}\,dx$$

$$\Big[4\ln(x+2) + 3(x+2)^{-1}\Big]_1^4$$

$$\left(4\ln 6 + \frac{1}{2}\right) - \left(4\ln 3 + 1\right)$$

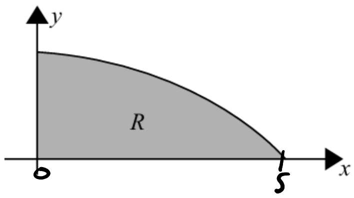
$$4\ln 6 - 4\ln 3 - \frac{1}{2}$$

$$4(\ln 6 - \ln 3) - \frac{1}{2}$$

$$\underline{\underline{4\ln 2 - \frac{1}{2}}}$$

66 The curve C with equation $y = \ln(6 - x)$ is shown on the diagram.

The area R is enclosed by C, the x-axis and the y-axis.



Use the trapezium rule with 6 ordinates to find an estimate for the area of R.

x	0	1	2	3	4	5
y	$\ln 6$	$\ln 5$	$\ln 4$	$\ln 3$	$\ln 2$	0

$$1\left(\frac{\ln 6}{2} + \ln 5 + \ln 4 + \ln 3 + \ln 2\right)$$

$\underline{\underline{5.68}}$ units^2

---

67 Find $\int \frac{4}{x^3} \ln x \, dx$

$u = \ln x \qquad \frac{dv}{dx} = 4x^{-3}$

$\frac{du}{dx} = \frac{1}{x} \qquad v = -2x^{-2}$

$$-2x^{-2} \ln x - \int -2x^{-3} \, dx$$

$$-2x^{-2} \ln x - x^{-2} + c$$

$$\underline{\underline{-\frac{2}{x^2} \ln x - \frac{1}{x^2} + c}}$$

68 A model for the growth of a population, P, can be expressed using the differential equation

$$\frac{dP}{dt} = 1000 - \frac{3}{5}P$$

(a) Given that $P = 0$ when $t = 0$, solve the differential equation to find P in terms of t.

(b) Sketch the graph of P against t.

$$\int \frac{1}{1000 - \frac{3}{5}P}\, dP = \int 1 \, dt$$

$$\int \frac{5}{5000 - 3P}\, dP = \int 1 \, dt$$

$$-\frac{5}{3}\ln(5000 - 3P) = t + c$$

$(0,0)$ $$c = -\frac{5}{3}\ln 5000$$

$$-\frac{5}{3}\ln(5000 - 3P) = t - \frac{5}{3}\ln 5000$$

$$\ln(5000 - 3P) = -\frac{3}{5}t - \ln 5000$$

$$5000 - 3P = e^{-\frac{3}{5}t + \ln 5000}$$

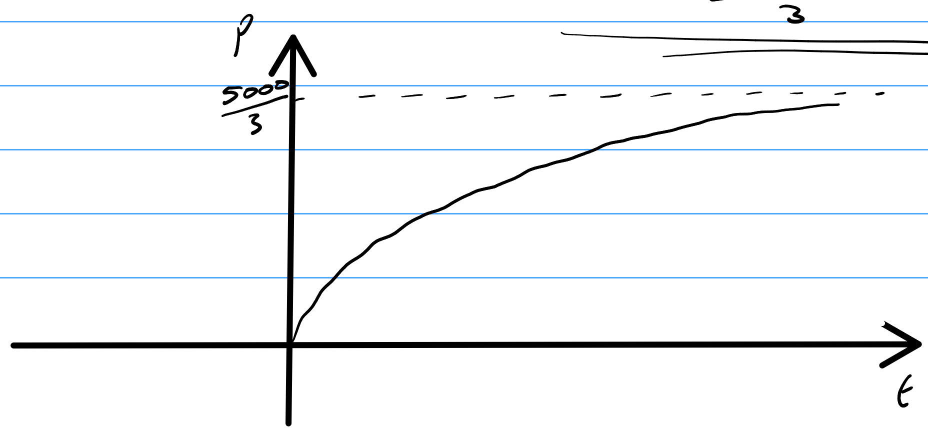
$$5000 - 3P = \frac{e^{\ln 5000}}{e^{3/5 t}}$$

$$5000 - 3P = \frac{5000}{e^{\frac{3}{5}t}}$$

$$3P = 5000 - \frac{5000}{e^{3/5 t}}$$

$$P = \frac{5000}{3} - \frac{5000}{3e^{3/5 t}}$$

b/



69 Use the substitution $x = u^2$ to find the exact value of $\int_1^4 \frac{dx}{(2+\sqrt{x})}$

when $x = 4$ $\quad u^2 = 2$ $\qquad \frac{dx}{du} = 2u$

$x = 1$ $\quad u^2 = 1$

$$\int_1^2 \frac{1}{2+u} \cdot 2u \, du$$

$$\int_1^2 \frac{2u}{2+u} \, du$$

$$\frac{2u}{2+u} = A + \frac{B}{2+u}$$

$$2u = A(2+u) + B$$

let $u=-2$ $\quad -4 = B$

let $u=0$ $\quad 0 = 2A - 4$

$$A = 2$$

$$\int_1 2 - \frac{4}{2+u} \, du$$

$$\left[2u - 4\ln(2+u) \right]_1^2$$

$4 - 4\ln 4 - (2 - 4\ln 3)$

$4 - 4\ln 4 - 2 + 4\ln 3$

$\underline{\underline{2 - 4\ln 4 + 4\ln 3}}$

70 The gradient of $y = \mathrm{f}(x)$ is given by the differential equation

$$(2x+1)^2 \frac{\mathrm{d}y}{\mathrm{d}x} - y^2 = 0$$

and the curve passes through the point (0, 1).

(a) Solve this differential equation to find $\mathrm{f}(x)$.

(b) Hence, suggest a suitable domain for $\mathrm{f}(x)$.

$$(2x+1)^2 \frac{dy}{dx} = y^2$$

$$\int \frac{1}{y^2}\,dy = \int \frac{1}{(2x+1)^2}\,dx$$

$$\int y^{-2}\,dy = \int (2x+1)^{-2}\,dx$$

$$-y^{-1} = -\frac{1}{2}(2x+1)^{-1} + c$$

$(0,1)$ $$-1 = -\frac{1}{2} + c$$

$$c = -\frac{1}{2}$$

$$-\frac{1}{y} = -\frac{1}{2(2x+1)} - \frac{1}{2}$$

$$\frac{1}{y} = \frac{1}{2(2x+1)} + \frac{1}{2}$$

$$\frac{1}{y} = \frac{1}{2(2x+1)} + \frac{2x+1}{2(2x+1)}$$

$$\frac{1}{y} = \frac{2x+2}{2(2x+1)}$$

$$y = \frac{2(2x+1)}{2x+2}$$

$$y = \frac{2x+1}{x+1}$$

b) $\underline{\underline{x \in \mathbb{R} \quad x \neq -1}}$

71 A tank is initially full of water. There is an open tap at the base of the tank which the water is draining out of. The height of water in the tank is h cm at time t seconds. The rate of change of the height of water is modelled as being directly proportional to the square root of the height of water.

When $t = 10$, $h = 36$ and, at this instant, the height is decreasing at a rate of $0.4\ \text{cms}^{-1}$

(a) Show that $\dfrac{dh}{dt} = -\dfrac{1}{15}\sqrt{h}$ **(2)**

(b) Find an expression for h in terms of t. **(4)**

(c) Work out how long, according to the model, it takes for the tank to empty. **(1)**

$$\frac{dh}{dt} = -k\sqrt{h}$$

when $t=10$, $h=36$ $\qquad \dfrac{dh}{dt} = -0.4$

$$-k\sqrt{h} = -0.4$$
$$-k\sqrt{36} = -0.4$$
$$-k = \frac{-0.4}{6}$$
$$k = \frac{1}{15}$$

$$\frac{dh}{dt} = -\frac{1}{15}\sqrt{h}$$

b/ $$\int \frac{1}{\sqrt{h}}\,dh = \int -\frac{1}{15}\,dt$$

$$\int h^{-\frac{1}{2}}\,dh = \int -\frac{1}{15}\,dt$$

$$2h^{\frac{1}{2}} = -\frac{1}{15}t + c$$

$$2(36)^{\frac{1}{2}} = -\frac{1}{15}(10) + c$$

$$12 = -\frac{2}{3} + c$$

$$c = \frac{38}{3}$$

$$2\sqrt{h} = -\frac{1}{15}t + \frac{38}{3}$$

$$\sqrt{h} = -\frac{1}{30}t + \frac{38}{6}$$

$$h = \left(-\frac{1}{30}t + \frac{19}{3}\right)^2$$

c/ $$0 = \left(-\frac{1}{30}t + \frac{19}{3}\right)^2$$

$$-\frac{1}{30}t + \frac{19}{3} = 0$$

$$\frac{1}{30}t = \frac{19}{3}$$

$$\underline{t = 190 \text{ seconds}}$$