A Level Maths: Integration

1 (a) Find: $\int \cos x \, dx$ **(1)**

(b) Find: $\int \text{cosec } 3x \cot 3x \, dx$ **(2)**

(c) Find: $\int 2 \sec^2 4x \, dx$ **(2)**

(Total for question 1 is 5 marks)

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2 (a) Express $\tan^2\theta$ in terms of $\sec^2\theta$ **(1)**

(b) Find: $\int \tan^2\theta \, d\theta$ **(3)**

(Total for question 2 is 4 marks)

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3 (a) Find: $\int \frac{\sin x}{\cos^2 x} \, dx$ **(3)**

(b) Find: $\int 2\text{cosec}^2 x + \sin x \, dx$ **(3)**

(Total for question 3 is 6 marks)

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4 (a) Find: $\int \frac{1}{\cos^2(3x+1)} \, dx$ **(3)**

(b) Find: $\int (\sec x + \tan x)^2 \, dx$ **(4)**

(Total for question 4 is 7 marks)

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5 Find: $\int_0^{\frac{\pi}{4}} \sin 2x \cos 2x \, dx$

(Total for question 5 is 5 marks)

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6 (a) Express $\sin^2 x$ in terms of $\cos 2x$ **(2)**

(b) Find: $\int \sin^2 x \, dx$ **(3)**

(Total for question 6 is 5 marks)

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7 Find: $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

(Total for question 7 is 6 marks)

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8 Use the substitution $u = x^2 + 4$ to find $\int \frac{2x}{x^2 + 4}\, dx$

(Total for question 8 is 5 marks)

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9 Use the substitution $u = \sin x$ to find $\int \sin^3 x \cos x \, dx$

(Total for question 9 is 5 marks)

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10 Use the substitution $u = x^2 + 2$ to find $\int 2x(x^2 + 2)^2 \, dx$

(Total for question 10 is 5 marks)

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11 Use the substitution $u = 1 + e^x$ to find $\int \frac{e^{3x}}{1 + e^x}\, dx$

(Total for question 11 is 7 marks)

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12 Use the substitution $u = x^3 - 4$ to find $\int_2^3 2x^2(x^3 - 4)^2 \, dx$

(Total for question 12 is 6 marks)

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13 Use the substitution $x = \sin u$ to find $\int_0^{0.5} \frac{1}{\sqrt{1 - x^2}}\, dx$

(Total for question 13 is 7 marks)

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14 Use the substitution $u = 1 + \cos x$ to find $\int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos x}\, dx$

(Total for question 14 is 7 marks)

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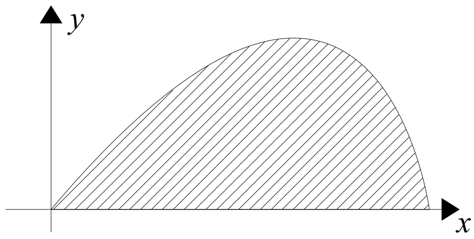
15 The curve C has the parametric equations

$$x = 5\cos t \qquad y = 3\sin 2t \qquad 0 \le t \le \frac{\pi}{2}$$



(a) Find the points where the curve meets the x axis **(3)**

(b) Find the area between the curve and the x axis **(7)**

(Total for question 15 is 10 marks)

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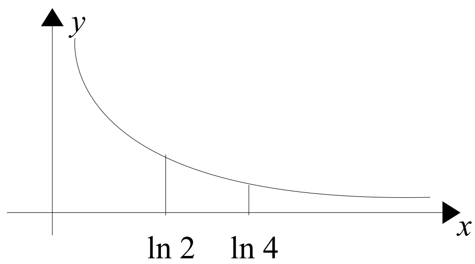
16 The curve C has the parametric equations

$$x = \ln(t+2) \qquad y = \frac{1}{t+1} \qquad t > -1$$



The finite region R between by the curve C and the x axis is bounded by the the lines with equations $x = \ln 2$ and $x = \ln 4$

(a) Show that the area of R is given by the integral $\int_0^2 \frac{1}{(t+1)(t+2)}\,dt$ **(4)**

(b) Hence find an exact value for this area **(6)**

(Total for question 16 is 10 marks)

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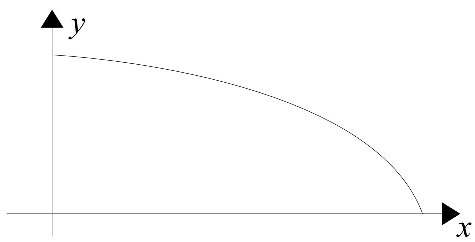
17 The curve C has the parametric equations

$$x = 3\cos 2t \qquad y = 3\sin t \qquad 0 \le t \le \frac{\pi}{4}$$



The finite region R between the curve C and the x axis.

(a) The point P is where the curve meets the x axis. Find the coordinates of P. **(2)**

(b) Show that the area of R is given by the integral $\int_0^{\frac{\pi}{4}} 36\sin^2 t\cos t\,dt$ **(5)**

(c) Hence find an exact value for the area **(5)**

(Total for question 17 is 12 marks)

18 Use integration by parts to find $\int x \sin x \quad dx$

(Total for question 18 is 4 marks)

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19 Use integration by parts to find $\int 2xe^x \ dx$

(Total for question 19 is 4 marks)

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20 Use integration by parts to find $\int x \sec^2 x \ dx$

(Total for question 20 is 4 marks)

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21 Use integration by parts to find $\int xe^{3x} \ dx$

(Total for question 21 is 4 marks)

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22 Use integration by parts to find the exact value of $\int_0^{\frac{\pi}{6}} 2x \cos x \quad dx$

(Total for question 22 is 6 marks)

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23 Use integration by parts, twice, to find $\int x^2 e^x \quad dx$

(Total for question 23 is 6 marks)

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24 Use integration by parts to find $\int \ln x \quad dx$

(Total for question 24 is 4 marks)

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25 Use integration by parts to find the exact value of $\int_1^2 x^2 \ln x \quad dx$

(Total for question 25 is 6 marks)

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26 Integrate with respect to x

(a) $(2x + 3)^2$ **(3)**

(b) $\frac{2}{(5x-1)^3}$ **(3)**

(Total for question 26 is 6 marks)

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27 Integrate with respect to x

(a) $\sqrt{5x-3}$ **(3)**

(b) e^{2x+3} **(3)**

(Total for question 27 is 6 marks)

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28 Integrate with respect to x

(a) e^{3-x} **(3)**

(b) $\frac{1}{2x+1}$ **(3)**

(Total for question 28 is 6 marks)

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29 Find: $\int_1^2 \frac{2}{3x+5}\, dx$

(Total for question 29 is 5 marks)

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30 Find: $\int_0^1 \frac{10}{(2x+1)^3}\, dx$

(Total for question 30 is 5 marks)

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31 (a) Express $\frac{3x}{(x+1)(x-2)}$ in partial fractions **(3)**

(b) Integrate $\frac{3x}{(x+1)(x-2)}$ with respect to x **(2)**

(Total for question 31 is 5 marks)

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32 Find the exact value of $\int_0^5 \frac{2x^2 + 3x + 7}{(x+1)^2(x+3)}\, dx$

(Total for question 32 is 8 marks)

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33 The annual rate of increase of a population is equal to 2% of the size of the population.
y is the population in millions and t is the time in years.

(a) Write down a differential equation for this relationship **(2)**

(b) Show that $y = \mathrm{A}e^{0.02t}$ where A is a constant **(4)**

(c) Given that the initial population is 2.5 million. Find the population after 10 years. **(2)**

(Total for question 33 is 8 marks)

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34 The rate of change of the temperature of a kettle of water (y) after it boils is directly proportional to the difference between the temperature of the water and the room temperature (20°C).

(a) Write down a differential equation for this relationship **(3)**

(b) Show that $y = 20 + \mathrm{A}e^{kt}$ where A and k are constants **(4)**

(c) Given that the initial temperature is 100℃ write down the value of A. **(1)**

(d) After 8 minutes the temperature is 60℃ show that $k = -\frac{1}{8}\ln 2$ **(3)**

(Total for question 34 is 11 marks)

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35 (a) Express $\dfrac{3x-3}{(x+1)(2x-1)}$ in partial fractions **(3)**

(b) Given that $x > 1$, find the general solution to the differential equation **(5)**

$$(x+1)(2x-1)\frac{dy}{dx} = y(3x-3)$$

(c) Hence find the particular solution to the differential equation that satisfies $y = 6$ at $x = 5$, giving your answer in the form $y = f(x)$ **(4)**

(Total for question 35 is 12 marks)

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36 Find the general solution to the differential equation $\dfrac{dy}{dx} = xy\sin x$

(Total for question 36 is 6 marks)

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37 Find the general solution to the differential equation $\dfrac{dy}{dx} = y^2\ln x$

(Total for question 37 is 6 marks)

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38 Find the solution to the differential equation $\dfrac{dy}{dx} = (y+1)^2$ given that when $y = 0$, $x = 2$

Give your answer in the form $y = f(x)$

(Total for question 38 is 6 marks)

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39 The height of a Ferris wheel above the ground, H metres is modelled by the differential equation:

$$\frac{dH}{dt} = \frac{H\sin(0.2t)}{5}$$

where t is the time, in minutes, from the start of the ride.

Given that the passenger is $\frac{48}{e}$ m above the ground at the start of the ride,

(a) Show that $H = 48e^{-\cos(0.2t)}$ **(5)**

(b) Find the time, to the nearest minute, it takes for the ride to reach its maximum height. **(2)**

(Total for question 39 is 7 marks)

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40 Show that $\int_0^3 \frac{5x}{\sqrt{x+1}}\, dx = \frac{40}{3}$

(Total for question 40 is 6 marks)

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41 The rate of decrease in temperature of water in a kettle t minutes after it switches off is given by:

$$\frac{d\theta}{dt} = -k(\theta - 20)$$

Where θ is the temperature of the water in degrees Celsius and k is a constant to be found.

Given that when t = 0, $\theta = 98$ and when t = 5, $\theta = 59$

Find a model for θ in the form $\theta = Ae^{kt} + B$, stating the values of A, B and k.

(Total for question 41 is 6 marks)

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42 The rate of increase of a population, P, t days after it is initially recorded is given by:

$$\frac{dP}{dt} = \frac{P}{5}$$

Given that the initial population was 50.

Find a model for P in the form $P = Ae^{kt}$

(Total for question 42 is 5 marks)

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43 After it rains a puddle has 40 cm^3 of water.

Two hours later the puddle has 30 cm^3 of water.

In a simple model the rate of decrease of the volume of water is inversely proportional to the square of the volume of water in the puddle.

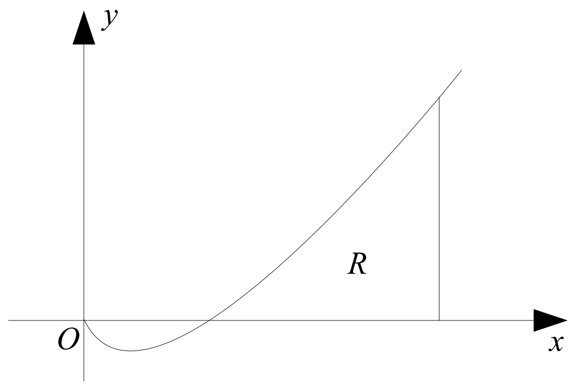
(a) Find an equation linking the volume of water in the puddle and time. **(5)**

(b) Hence find the total time taken for the water in the puddle to fully evaporate.
Give your answer in hours and minutes, to the nearest minute. **(3)**

(c) Suggest a limitation of the model. **(1)**

(Total for question 43 is 9 marks)

44



The diagram shows the curve $y = x \ln x$, $x > 0$

The region R is bounded by the curve, the line $x = 4$ and the x-axis.

Show that the exact area of $R = A\ln 2 + B$ where A and B are rational numbers to be found.

(Total for question 44 is 7 marks)

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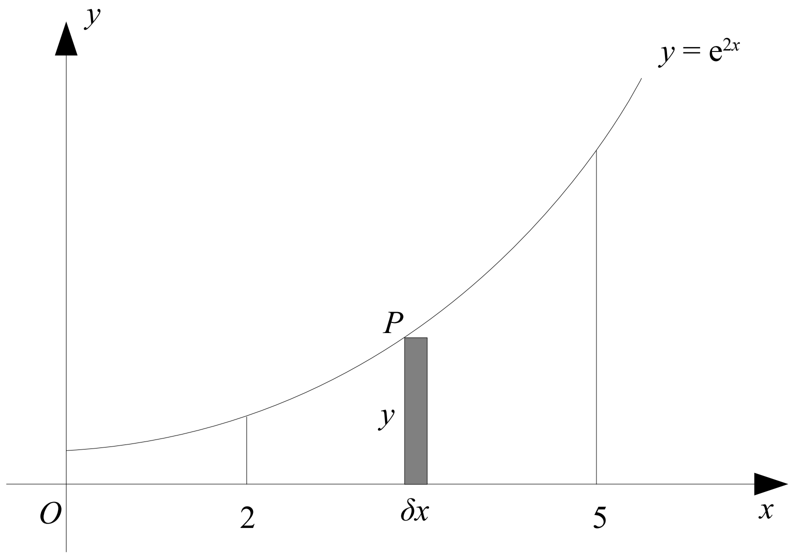
45 $y = (x^2 - 3)^5$

(a) Find $\frac{dy}{dx}$ **(2)**

(b) Hence find $\int 2x(x^2 - 3)^4 \, dx$ **(2)**

(Total for question 45 is 4 marks)

46



The diagram shows the curve $y = e^{2x}$, $x > 0$

The point $P(x, y)$ lies on the curve.
The rectangle, shown shaded on the diagram, has height y and width δx

Calculate

$$\lim_{\delta x \to 0} \sum_{x=2}^{5} e^{2x} \delta x$$

(Total for question 46 is 3 marks)

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47 Use the substitution $u = 2 - \sqrt{x}$ to show that

$$\int \frac{1}{2 - \sqrt{x}} = -4 \ln|2 - \sqrt{x}| - 2\sqrt{x} + k$$

where k is a constant.

(Total for question 47 is 6 marks)

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48 (a) Use the substitution $x = u^2 + 1$ to show that

$$\int_2^{10} \frac{2\,dx}{(x-1)(2+\sqrt{x-1})} = \int_q^p \frac{4\,du}{u(u+2)}$$

where p and q are positive constants to be found. **(4)**

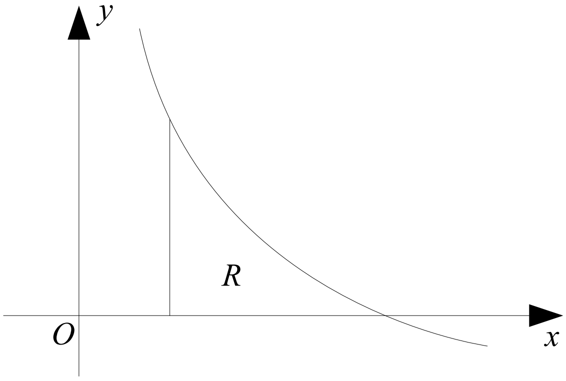
(b) Hence, using algebraic integration, show that

$$\int_2^{10} \frac{2\,dx}{(x-1)(2+\sqrt{x-1})} = \ln a$$

where a is a rational constant to be found. **(6)**

(Total for question 48 is 10 marks)

49



The curve C with equation $y = \dfrac{15 - x}{(2x - 3)(x + 1)}$

The region R is bounded by the curve C, the x-axis and the line with equation $x = 3$

Show that the exact value of the area of R is $a \ln 2 + b \ln 3$, where a and b are rational constants to be found.

(Total for question 49 is 8 marks)

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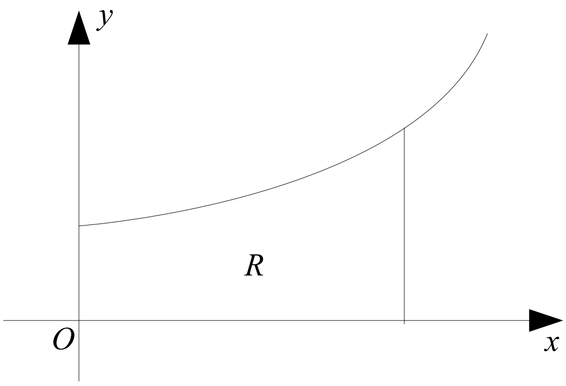
50 (a) Complete the table below, giving values of $\sqrt{2^x + 1}$ to 3 decimal places.

x	0	0.5	1	1.5	2	2.5	3
$\sqrt{2^x + 1}$	1.414	1.554	1.732	1.957			

(2)

The region R which is bounded by the curve with equation $y = \sqrt{2^x + 1}$, the x-axis and the lines $x = 0$ and $x = 3$



(b) Use the trapezium rule to find an approximation for the area of R. **(4)**

(c) State, giving a reason, whether your approximation in part (b) is an overestimate or an underestimate for the area of R. **(2)**

(Total for question 50 is 8 marks)

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51 A spherical balloon is being inflated.

At time t seconds the balloon has a radius r cm and a volume V cm^3.

The volume of the balloon is modelled as increasing at a constant rate.

(a) Show that

$$\frac{dr}{dt} = \frac{k}{r^2}$$

where k is a positive constant.

(3)

Given that the balloon is initially empty and after 5 seconds the radius of the balloon is 4 cm.

(b) Solve the differential equation to find an equation linking r and t. **(5)**

(c) Suggest a limitation of the model. **(1)**

(Total for question 51 is 9 marks)

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52 The table below shows corresponding values of x and y for $y = \sqrt{\frac{2+x}{1+x}}$

The values of y are given to 4 significant figures.

x	0	0.5	1	1.5	2
y	1.414	1.291	1.225	1.183	1.155

(a) Use the trapezium rule, with all the values of y in the table, to find an estimate for

$$\int_0^2 \sqrt{\frac{2+x}{1+x}}\, dx$$

giving your answer to 3 significant figures.

(3)

(b) Using your answer to part (a), deduce an estimate for $\int_0^2 \sqrt{\frac{8+4x}{1+x}}\, dx$ **(1)**

(Total for question 52 is 4 marks)

53 (a) Given that $\dfrac{2x^2 - x + 1}{x + 1} \equiv Ax + B + \dfrac{C}{x + 1}$ $\quad x \neq -1$

find the values of the constants A, B and C. **(3)**

(b) Hence, using algebraic integration, find the exact value of

$$\int_0^7 \frac{2x^2 - x + 1}{x + 1} \, dx$$

giving your answer in the form $a + b \ln 2$ where a and b are integers to be found. **(4)**

(Total for question 53 is 7 marks)

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54 A curve C has equation $y = f(x)$

Given that

- $f'(x) = 6x^2 + ax + 7$ where a is a constant
- the y intercept of C is -20
- $(x + 4)$ is a factor of $f(x)$

find, in simplest form, $f(x)$

(Total for question 54 is 6 marks)

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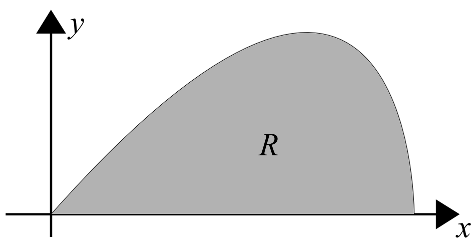
55 The curve C has the parametric equations

$x = 2 \sin t \qquad y = 3 \sin 2t \qquad 0 \leq t \leq \dfrac{\pi}{2}$



(a) Show that the area of R is given by $\displaystyle\int_0^{\frac{\pi}{2}} 12 \sin t \cos^2 t \; dt$ **(3)**

(b) Hence show, by algebraic integration, that the area of R is exactly 4 **(3)**

(Total for question 55 is 6 marks)

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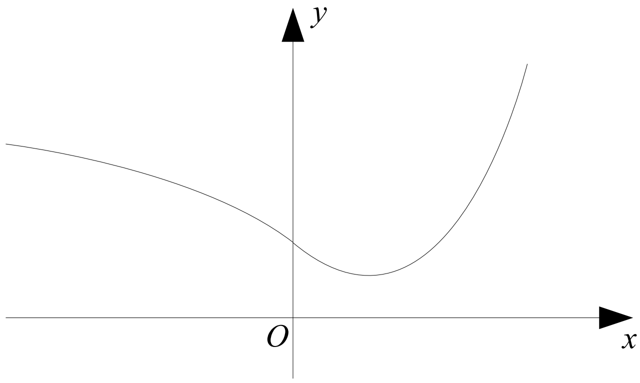
56 The curve C with equation $y = \mathrm{f}(x)$ is shown on the diagram.



Given the minimum point of C is $(1, \mathrm{e})$ and $\mathrm{f}'(x) = 2(x - 1)\mathrm{e}^x$ find an equation for $\mathrm{f}(x)$.

(Total for question 56 is 6 marks)

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57 (a) Prove the identity $\dfrac{\sin 2x}{1 + \cot^2 x} \equiv 2\sin^3 x \cos x$ **(3)**

(b) Hence, find $\displaystyle\int \frac{4\sin 4\theta}{1 + \cot^2 2\theta}\,\mathrm{d}\theta$

(5)

(Total for question 57 is 8 marks)

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58 Show that $\displaystyle\int_0^6 \frac{4x^2}{x + 2}\,\mathrm{d}x = A + B\ln 2$

where A and B are integers to be found.

(Total for question 58 is 5 marks)

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59 (a) $y = \mathrm{e}^x(\sin x - \cos x)$

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$

Simplify your answer.

(3)

(b) Hence, find $\displaystyle\int \mathrm{e}^x \sin x\,\mathrm{d}x$

(2)

(Total for question 59 is 5 marks)

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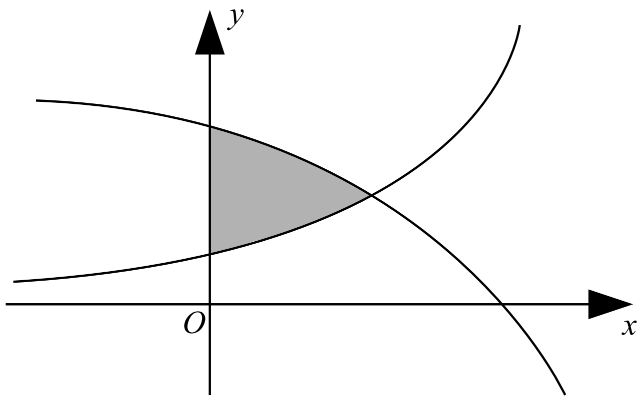
60 The region enclosed by the y-axis, $y = e^{2x}$ and $y = 12 - e^{x}$ is shown on the diagram.



Find the exact area of the shaded region.

(Total for question 60 is 10 marks)

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61 Use integration by substitution to show that $\int_1^2 x\sqrt{5x-1}\,dx = \frac{1456}{375}$

(Total for question 61 is 6 marks)

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62 (a) Use the trapezium rule, with four strips each of width 0.5, to find an estimate for

$$\int_1^3 x\sqrt{e^x}\,dx$$ **(3)**

(b) Explain how the trapezium rule could be used to obtain a more accurate estimate. **(1)**

(Total for question 62 is 4 marks)

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63 Show that $\int_0^{\frac{\pi}{2}} \frac{4\sin 2x}{3-\cos(2x)}\,dx = \ln 4$

(Total for question 63 is 6 marks)

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64 Solve the differential equation

$$\frac{dy}{dx} = \frac{\ln x}{2x^2 y} \qquad \text{for } x > 0$$

Given $x = 1$ when $y = 1$

Write your answer in the form $y^2 = f(x)$

(Total for question 64 is 7 marks)

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65 (a) Express $\frac{4x+5}{(x+2)^2}$ in the form $\frac{A}{(x+2)} + \frac{B}{(x+2)^2}$ **(3)**

(b) Show that $\int_1^4 \frac{4x+5}{(x+2)^2} dx = a + b \ln 2$

where a and b are rational numbers. **(5)**

(Total for question 65 is 8 marks)

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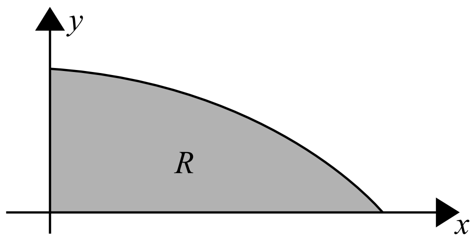
66 The curve C with equation $y = \ln(6 - x)$ is shown on the diagram.

The area R is enclosed by C, the x-axis and the y-axis.



Use the trapezium rule with 6 ordinates to find an estimate for the area of R.

(Total for question 66 is 5 marks)

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67 Find $\int \frac{4}{x^3} \ln x \, dx$

(Total for question 67 is 4 marks)

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68 A model for the growth of a population, P, can be expressed using the differential equation

$$\frac{dP}{dt} = 1000 - \frac{3}{5}P$$

(a) Given that $P = 0$ when $t = 0$, solve the differential equation to find P in terms of t. **(7)**

(b) Sketch the graph of P against t. **(2)**

(Total for question 68 is 9 marks)

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69 Use the substitution $x = u^2$ to find the exact value of $\displaystyle\int_1^4 \frac{dx}{(2+\sqrt{x})}$

(Total for question 69 is 7 marks)

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70 The gradient of $y = f(x)$ is given by the differential equation

$$(2x+1)^2 \frac{dy}{dx} - y^2 = 0$$

and the curve passes through the point (0, 1).

(a) Solve this differential equation to find $f(x)$. **(6)**

(b) Hence, suggest a suitable domain for $f(x)$. **(1)**

(Total for question 70 is 7 marks)

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71 A tank is initially full of water. There is an open tap at the base of the tank which the water is draining out of. The height of water in the tank is h cm at time t seconds. The rate of change of the height of water is modelled as being directly proportional to the square root of the height of water.

When $t = 10$, $h = 36$ and, at this instant, the height is decreasing at a rate of 0.4 cms^{-1}

(a) Show that $\dfrac{dh}{dt} = -\dfrac{1}{15}\sqrt{h}$ **(2)**

(b) Find an expression for h in terms of t. **(4)**

(c) Work out how long, according to the model, it takes for the tank to empty. **(1)**

(Total for question 71 is 7 marks)
