1 $f(x) = \ln x + e^x$

(a) Find $f'(x)$
(b) Find $f''(x)$

a) $f'(x) = \frac{1}{x} + e^x$

b) $f'(x) = x^{-1} + e^x$
$f''(x) = -x^{-2} + e^x$

---

2 Differentiate with respect to x,

(a) $2x^3 + e^{4x}$
(b) $(x^2 + 5)^3$

a) $6x^2 + 4e^{4x}$

b) $3(x^2+5)^2 \times 2x$
$6x(x^2+5)^2$

---

3 Differentiate with respect to x,

(a) $x + \dfrac{5}{x^2+4}$

(b) $8 + e^{x^2}$

a) $x + 5(x^2+4)^{-1}$

$1 - 5(x^2+4)^{-2}(2x)$

$1 - 10x(x^2+4)^{-2}$

b) $2xe^{x^2}$

4 The point P lies on the curve with equation $y = 2 + \ln(3 - 2x)$ with x coordinate 1

Find an equation to the tangent to the curve at the point P.

when $x = 1$ $\quad y = 2 + \ln 1$ $\quad (1, 2)$

$= 2$

$$\frac{dy}{dx} = \frac{1}{3-2x}(-2)$$

$$= \frac{-2}{3-2x} \quad \text{when } x = 1 \quad \frac{dy}{dx} = \frac{-2}{3-2} = -2$$

$$y - 2 = -2(x - 1)$$

$$\underline{\underline{y = -2x + 4}}$$

5 The point P lies on the curve with equation $y = \dfrac{3}{2x+1}$ with x coordinate 1

(a) Find an equation to the normal to the curve at the point P.

The normal intersects the curve again at the point Q.

(b) Find the exact coordinates of Q.

a/ $\quad y = 3(2x+1)^{-1}$

$$\frac{dy}{dx} = -3(2x+1)^{-2}(2)$$

$$= \underline{-6(2x+1)^{-2}}$$

when $x = 1$ $\quad \frac{dy}{dx} = -6(3)^{-2} = -\frac{2}{3}$

$$y = \frac{3}{2(1)+1} = 1 \quad (1, 1)$$

gradient of normal $= \frac{3}{2}$

$$y - 1 = \frac{3}{2}(x - 1)$$

$y = \frac{3}{2}x - \frac{3}{2} + 1$

$= \frac{3}{2}x - \frac{1}{2}$

b/ $y = \frac{3}{2}x - \frac{1}{2}$ $\qquad$ $y = \frac{3}{2x+1}$

$\frac{3}{2}x - \frac{1}{2} = \frac{3}{2x+1}$

$3x - 1 = \frac{6}{2x+1}$

$(3x-1)(2x+1) = 6$

$6x^2 + 3x - 2x - 1 = 6$

$6x^2 + x - 7 = 0$

$x = 1$ $\qquad$ $x = \frac{-7}{6}$

$y = \frac{3}{2}\left(-\frac{7}{6}\right) - \frac{1}{2}$

$= -\frac{9}{4}$

$\left(-\frac{7}{6}, -\frac{9}{4}\right)$

6 The point P lies on the curve with equation $y = \dfrac{2}{\sqrt{2x+1}}$ with x coordinate 4

(a) Find an equation to the tangent to the curve at the point P.

The tangent intersects the x axis at the point A and the y axis at the point B.

(b) Find the exact area of the triangle AOB, where O is the origin.

a/ when $x = 4$ $\quad y = \dfrac{2}{3}$

$$y = 2(2x+1)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -2(2x+1)^{-\frac{3}{2}}$$

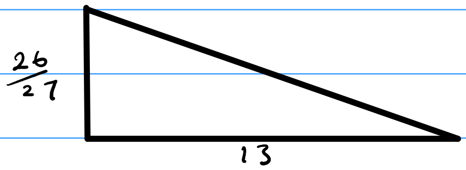
when $x = 4$ $\quad \dfrac{dy}{dx} = \dfrac{-2}{27}$

$$\underline{y - \frac{2}{3} = \frac{-2}{27}(x-4)}$$

b/ $y = -\dfrac{2}{27}x + \dfrac{26}{27}$

crosses y at $\dfrac{26}{27}$

crosses x when $y = 0$ $\quad 0 = \dfrac{-2}{27}x + \dfrac{26}{27}$ $\quad x = 13$



Area $= \frac{1}{2}(13)\left(\frac{26}{27}\right) = \dfrac{169}{27}$ units^2

7 Differentiate with respect to x

(a) $2xe^x$
(b) $3x^2 \ln 2x$

a/ $u = 2x$ $v = e^x$

$\frac{du}{dx} = 2$ $\frac{dv}{dx} = e^x$

$2e^x + 2xe^x$

b/ $u = 3x^2$ $v = \ln 2x$

$\frac{du}{dx} = 6x$ $\frac{dv}{dx} = \frac{1}{x}$

$3x + 6x \ln 2x$

---

8 Differentiate with respect to x,

(a) $x^2\sqrt{2x+1}$
(b) $x \ln(x+1)$

a/ $u = x^2$ $v = (2x+1)^{\frac{1}{2}}$

$\frac{du}{dx} = 2x$ $\frac{dv}{dx} = (2x+1)^{-\frac{1}{2}}$

$x^2(2x+1)^{-\frac{1}{2}} + 2x(2x+1)^{\frac{1}{2}}$

b/ $u = x$ $v = \ln(x+1)$

$\frac{du}{dx} = 1$ $\frac{dv}{dx} = \frac{1}{x+1}$

$\frac{x}{x+1} + \ln(x+1)$

9 Differentiate with respect to x,

(a) $(x + 5)(x + 1)^3$
(b) $2 + x^3 \ln(2x + 1)$

a/

$u = x + 5 \qquad v = (x+1)^3$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 3(x+1)^2$

$(x+1)^3 + 3(x+5)(x+1)^2$

b/

$u = x^3 \qquad v = \ln(2x+1)$

$\frac{du}{dx} = 3x^2 \qquad \frac{dv}{dx} = \frac{2}{2x+1}$

$\frac{2x^3}{2x+1} + 3x^2 \ln(2x+1)$

---

10 The point P lies on the curve with equation $y = (3x - 1)\ln(2 - x)$ with x coordinate 1.

Find an equation to the tangent to the curve at the point P.

when $x = 1 \quad y = 0$

$u = 3x - 1 \qquad v = \ln(2 - x)$

$\frac{du}{dx} = 3 \qquad \frac{dv}{dx} = \frac{-1}{2-x}$

$\frac{dy}{dx} = \frac{-(3x-1)}{2-x} + 3\ln(2-x)$

when $x = 1 \quad \frac{dy}{dx} = -2$

$y - 0 = -2(x - 1)$

$y = -2x + 2$

11 Find the coordinates of the stationary points of the curve $y = x(x-3)^3$ and determine the nature of the stationary points.

$u = x \qquad v = (x-3)^3$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 3(x-3)^2$

$\frac{dy}{dx} = (x-3)^3 + 3x(x-3)^2$

stationary points where $\frac{dy}{dx} = 0$

$$(x-3)^3 + 3x(x-3)^2 = 0$$
$$(x-3)^2(x-3+3x) = 0$$
$$(x-3)^2(4x-3) = 0$$

$x = 3 \qquad x = \frac{3}{4}$

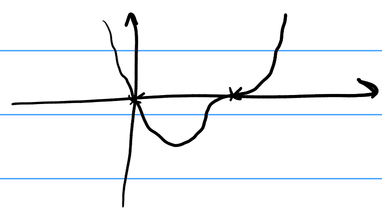
$y = 0 \qquad y = \frac{-2187}{256}$

$(3, 0) \qquad \left(\frac{3}{4}, \frac{-2187}{256}\right)$

$\frac{dy}{dx} = (x-3)^2(4x-3)$

$u = (x-3)^2 \qquad v = (4x-3)$

$\frac{du}{dx} = 2(x-3) \quad \frac{dv}{dx} = 4$



$\frac{d^2y}{dx^2} = 4(x-3)^2 + 2(x-3)(4x-3)$

when $x = 3$ $\frac{d^2y}{dx^2} = 0$ when $x = \frac{3}{4}$ $\frac{d^2y}{dx^2} = \frac{81}{4}$

when $x = 2.9$ $\frac{dy}{dx} = 0.086$ $\frac{d^2y}{dx^2} > 0$ $\therefore$ minimum at $\left(\frac{3}{4}, \frac{-2187}{256}\right)$

when $x = 3.1$ $\frac{dy}{dx} = 0.094$

$\therefore$ point of inflection at $(3, 0)$

12 The point P lies on the curve with equation $y = x\sqrt{x-1}$ with x coordinate 5

Find an equation to the normal to the curve at the point P.

$$y = x(x-1)^{\frac{1}{2}}$$

$$u = x \qquad v = (x-1)^{\frac{1}{2}}$$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \frac{1}{2}(x-1)^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = (x-1)^{\frac{1}{2}} + \frac{1}{2}x(x-1)^{-\frac{1}{2}}$$

when $x = 5$ $\quad \frac{dy}{dx} = \frac{13}{4}$

$\therefore \; m = -\frac{4}{13}$

when $x = 5$ $\quad y = 10$

$$y - 10 = -\frac{4}{13}(x-5)$$

$$\left(\text{or} \quad y = \frac{-4}{13}x + \frac{150}{13}\right)$$

---

13 The point P lies on the curve with equation $y = xe^{x^2}$ with x coordinate 1

(a) Find an equation to the tangent to the curve at the point P.

The tangent intersects the x axis at the point A and the y axis at the point B.

(b) Find the exact area of the triangle AOB, where O is the origin.

$$u = x \qquad v = e^{x^2}$$

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 2xe^{x^2}$$

$$\frac{dy}{dx} = 2x^2e^{x^2} + e^{x^2}$$

when $x = 1$ $\quad \frac{dy}{dx} = 2e + e$

$$= 3e$$

when $x = 1$ $\quad y = e$

$$(y-e) = 3e(x-1)$$

$$y = 3ex - 3e + e$$

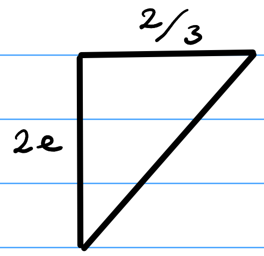
$$\underline{y = 3ex - 2e}$$

crosses y at $-2e$

crosses x when $0 = 3ex - 2e$

$$0 = 3x - 2$$

$$x = 2/3$$



Area $= \frac{1}{2} \cdot 2e \cdot \frac{2}{3}$

$$= \underline{\frac{2}{3} e \text{ units}^2}$$

---

14 Differentiate with respect to x

(a) $\dfrac{e^x}{2x+1}$

(b) $\dfrac{\ln(x^2+1)}{3x+2}$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

a/ $u = e^x$ $\qquad v = 2x+1$

$\frac{du}{dx} = e^x$ $\qquad \frac{dv}{dx} = 2$

$$\frac{dy}{dx} = \frac{(2x+1)e^x - 2e^x}{(2x+1)^2}$$

b/ $u = \ln(x^2+1)$ $\qquad v = 3x+2$

$\frac{du}{dx} = \frac{2x}{x^2+1}$ $\qquad \frac{dv}{dx} = 3$

$$\frac{dy}{dx} = \frac{\frac{(3x+2)(2x)}{x^2+1} - 3\ln(x^2+1)}{(3x+2)^2}$$

$$= \frac{2x(3x+2) - 3(x^2+1)\ln(x^2+1)}{(x^2+1)(3x+2)^2}$$

15 The point P lies on the curve with equation $y = \frac{3x}{2x-1}$ with x coordinate 1.

Find an equation to the tangent to the curve at the point P.

$u = 3x$ $\quad\quad$ $v = 2x - 1$

$\frac{du}{dx} = 3$ $\quad\quad$ $\frac{dv}{dx} = 2$

$$\frac{dy}{dx} = \frac{3(2x-1) - 2(3x)}{(2x-1)^2}$$

$$= \frac{6x - 3 - 6x}{(2x-1)^2}$$

$$= \frac{-3}{(2x-1)^2}$$

when $x = 1$ $\quad\quad$ $\frac{dy}{dx} = -3$

when $x = 1$ $\quad\quad$ $y = 3$

$$y - 3 = -3(x-1)$$

$$y = -3x + 6$$

16 The point P lies on the curve with equation $y = \dfrac{e^x + 1}{e^x + 3}$ with x coordinate 0.

Find an equation to the normal to the curve at the point P.

when $x = 0$ $\quad y = \frac{1}{2}$

$u = e^x + 1 \qquad v = e^x + 3$

$\frac{du}{dx} = e^x \qquad \frac{dv}{dx} = e^x$

$$\frac{dy}{dx} = \frac{e^x(e^x + 3) - e^x(e^x + 1)}{(e^x + 3)^2}$$

when $x = 0$ $\quad \frac{dy}{dx} = \frac{1}{8}$

$\therefore \; m = -8$

$(y - \frac{1}{2}) = -8(x - 0)$

$\underline{y = -8x + \frac{1}{2}}$

17 Find the coordinates of the stationary points of the curve $y = \frac{x}{9+x^2}$ and determine the nature of the stationary points.

$u = x \qquad v = 9 + x^2$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{9 + x^2 - 2x^2}{(9+x^2)^2}$$

$$= \frac{9 - x^2}{(9+x^2)^2}$$

stationary points where $\frac{9 - x^2}{(9+x^2)^2} = 0$

$$9 - x^2 = 0$$

$$(3+x)(3-x) = 0$$

$$x = -3 \quad x = 3$$

$u = 9 - x^2 \qquad v = (9 + x^2)^2$

$\frac{du}{dx} = -2x \qquad \frac{dv}{dx} = 4x(9 + x^2)$

$$\frac{d^2y}{dx^2} = \frac{-2x(9+x^2)^2 - 4x(9+x^2)(9-x^2)}{(9+x^2)^4}$$

when $x = -3$ $\frac{d^2y}{dx^2} = \frac{1}{54}$ when $x = 3$ $\frac{d^2y}{dx^2} = -\frac{1}{54}$

$\frac{d^2y}{dx^2} > 0$ $\therefore$ MINIMUM $\qquad$ $\frac{d^2y}{dx^2} < 0$ $\therefore$ MAXIMUM

when $x = -3$ $y = -\frac{1}{6}$ $\qquad$ when $x = 3$ $y = \frac{1}{6}$

Min. at $\left(-3, -\frac{1}{6}\right)$ $\qquad$ Max at $\left(3, \frac{1}{6}\right)$

18 $$f(x) = \frac{x-5}{2x+3} + \frac{2x+4}{2x^2+7x+6}$$

(a) Express f(x) as a fraction in its simplest form.

(b) Hence find f '(x) in its simplest form

a/
$$\frac{x-5}{2x+3} + \frac{2(x+2)}{(2x+3)(x+2)}$$

$$\frac{x-5}{2x+3} + \frac{2}{2x+3}$$

$$\frac{x-3}{2x+3}$$

b/
$u = x-3 \qquad v = 2x+3$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 2$

$$f'(x) = \frac{2x+3-2(x-3)}{(2x+3)^2}$$

$$= \frac{2x+3-2x+6}{(2x+3)^2}$$

$$= \frac{9}{(2x+3)^2}$$

19

$$f(x) = \frac{4x+1}{2x+3} - \frac{4x-6}{4x^2-9}$$

(a) Express f(x) as a fraction in its simplest form.

(b) Hence find f '(x) in its simplest form

a/

$$\frac{4x+1}{2x+3} - \frac{2(2x-3)}{(2x+3)(2x-3)}$$

$$\frac{4x+1}{2x+3} - \frac{2}{2x+3}$$

$$\frac{4x-1}{2x+3}$$

b/

$$u = 4x-1 \qquad v = 2x+3$$

$$\frac{du}{dx} = 4 \qquad \frac{dv}{dx} = 2$$

$$f'(x) = \frac{4(2x+3) - 2(4x-1)}{(2x+3)^2}$$

$$= \frac{8x+12-8x+2}{(2x+3)^2}$$

$$= \frac{14}{(2x+3)^2}$$

20 Use the derivatives of sin(x) and cos(x) to show that:

(a) $\frac{d}{dx}(\tan x) = \sec^2 x$

(b) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(c) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(d) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

a/ $\tan x = \frac{\sin x}{\cos x}$

$u = \sin x \qquad v = \cos x$

$\frac{du}{dx} = \cos x \qquad \frac{dv}{dx} = -\sin x$

$$\frac{d \tan x}{dx} = \frac{\cos^2 x - -\sin^2 x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}$$

$$= \underline{\underline{\sec^2 x}}$$

b/ $\sec x = \frac{1}{\cos x}$

$u = 1 \qquad v = \cos x$

$\frac{du}{dx} = 0 \qquad \frac{dv}{dx} = -\sin x$

$$\frac{d \sec x}{dx} = \frac{0 \cos x - -\sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x}$$

$$\frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$$

$$\underline{\underline{\sec x \tan x}}$$

c/

$$\cot x = \frac{\cos x}{\sin x}$$

$u = \cos x$ $\qquad$ $v = \sin x$

$\frac{du}{dx} = -\sin x$ $\qquad$ $\frac{dv}{dx} = \cos x$

$$\frac{d \cot x}{dx} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}$$

$$= \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x}$$

$$= -\frac{1}{\sin^2 x}$$

$$= \underline{\underline{-\operatorname{cosec}^2 x}}$$

d/ $\operatorname{cosec} x = \frac{1}{\sin x}$

$u = 1$ $\qquad$ $v = \sin x$

$\frac{du}{dx} = 0$ $\qquad$ $\frac{dv}{dx} = \cos x$

$$\frac{d \operatorname{cosec} x}{dx} = \frac{0 - \cos x}{\sin^2 x}$$

$$= \frac{-\cos x}{\sin^2 x}$$

$$= -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x}$$

$$= \underline{\underline{-\operatorname{cosec} x \cot x}}$$

21 Differentiate with respect to x,

(a) $x^2 \cos 2x$
(b) $3 \sin (2x + 1)$

a/ $u = x^2$ $v = \cos 2x$

$\frac{du}{dx} = 2x$ $\frac{dv}{dx} = -2 \sin 2x$

$$\underline{\underline{2x \cos 2x - 2x^2 \sin 2x}}$$

b/ $3 \cos(2x+1) \times 2x$

$\underline{\underline{6x \cos(2x+1)}}$

---

22 Differentiate with respect to x,

(a) $e^{3x} (\cos 2x + \sin x)$
(b) $\ln (\sin x)$

a/ $u = e^{3x}$ $v = \cos 2x + \sin x$

$\frac{du}{dx} = 3e^{3x}$ $\frac{dv}{dx} = -2 \sin 2x + \cos x$

$$\underline{\underline{3e^{3x} (\cos 2x + \sin x) + e^{3x}(-2 \sin 2x + \cos x)}}$$

b/ $\frac{1}{\sin x} \times \cos x$

$\underline{\underline{\cot x}}$

23 The curve C has the equation $x = 2\tan y$

(a) Find $\frac{dx}{dy}$ in terms of y

(b) Hence find $\frac{dy}{dx}$ in terms of x

a/ $\frac{dx}{dy} = 2\sec^2 y$

b/ $\frac{dy}{dx} = \frac{1}{2\sec^2 y}$ $\quad 1 + \tan^2 y = \sec^2 y$

$= \frac{1}{2 + 2\tan^2 y}$ $\quad \frac{x}{2} = \tan y$

$= \frac{1}{2 + \frac{x^2}{2}}$ $\quad \frac{x^2}{4} = \tan^2 y$

$= \frac{2}{4 + x^2}$ $\quad \frac{x^2}{2} = 2\tan^2 y$

---

24 The point P lies on the curve with equation $y = \text{cosec}\, x + \cos 2x$ with x coordinate $\frac{\pi}{4}$

Find an equation to the tangent to the curve at the point P.

$\frac{dy}{dx} = -\text{cosec}\, x \cot x - 2\sin 2x$

$= \frac{-\cos x}{\sin^2 x} - 2\sin 2x$

when $\frac{\pi}{4}$ $\quad \frac{dy}{dx} = -2 - \sqrt{2}$

when $x = \frac{\pi}{4}$ $\quad y = \sqrt{2}$

$y - \sqrt{2} = (-2 - \sqrt{2})(x - \frac{\pi}{4})$

25 The point P lies on the curve with equation $y = \sec 2x$ with x coordinate $\frac{\pi}{6}$

Find an equation to the normal to the curve at the point P.

$$\frac{dy}{dx} = 2\sec 2x \tan 2x$$

when $x = \frac{\pi}{6}$ $\quad \frac{dy}{dx} = 4\sqrt{3}$ $\quad \therefore$ gradient of normal $= -\frac{\sqrt{3}}{12}$

when $x = \frac{\pi}{6}$ $\quad y = 2$

$$y - 2 = -\frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)$$

26 A curve has the equation $2x^2 - 3xy + y^2 = 12$:

(a) Find an expression for $\frac{dy}{dx}$

(b) Find an equation for the normal to the curve at the point (1, -2)

$u = -3x \qquad v = y$

$\frac{du}{dx} = -3 \qquad \frac{dv}{dx} = \frac{dy}{dx}$

a/ $$4x - 3x\frac{dy}{dx} - 3y + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} - 3x\frac{dy}{dx} = 3y - 4x$$

$$\frac{dy}{dx}(2y - 3x) = 3y - 4x$$

$$\frac{dy}{dx} = \frac{3y - 4x}{2y - 3x}$$

b/ at $(1, -2)$ $\quad \frac{dy}{dx} = \frac{10}{7}$ $\quad \therefore$ gradient of normal $= -\frac{7}{10}$

$$y + 2 = -\frac{7}{10}(x - 1)$$

$$y = -\frac{7}{10}x - \frac{13}{10}$$

27 A curve has the equation $3x^2 + xy + y^2 = 20$

The gradient of the tangent to the curve is $\frac{4}{3}$ at the points P and Q.

(a) Show that $2x + y = 0$ at P and Q.

(b) Find the coordinates of P and Q.

a/ $6x + x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0$

$u = x \quad v = y$

$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{4}{3}$

$6x + \frac{4}{3}x + y + \frac{8}{3}y = 0$

$\frac{22}{3}x + \frac{11}{3}y = 0$

$22x + 11y = 0$

$\underline{2x + y = 0}$

b/ $3x^2 + xy + y^2 = 20 \qquad y = -2x$

$3x^2 + x(-2x) + (-2x)^2 = 20$

$3x^2 - 2x^2 + 4x^2 = 20$

$5x^2 = 20$

$x^2 = 4$

$x = \pm 2$

$x = 2 \quad y = -4 \qquad x = -2 \quad y = 4$

$\underline{\underline{(2, -4)}} \qquad \underline{\underline{(-2, 4)}}$

28 A curve has the equation $x^2 + 4xy - x + y^2 = 35$:

(a) Find an expression for $\frac{dy}{dx}$

(b) Find an equation for the tangent to the curve at the point P (2, 3)

a/ $2x + 4y + 4x\frac{dy}{dx} - 1 + 2y\frac{dy}{dx} = 0$

$u = 4x \quad v = y$

$\frac{du}{dx} = 4 \quad \frac{dv}{dx} = \frac{dy}{dx}$

$\frac{dy}{dx}(4x + 2y) = 1 - 2x - 4y$

$$\frac{dy}{dx} = \frac{1 - 2x - 4y}{4x + 2y}$$

b/ at $(2, 3)$ $\quad \frac{dy}{dx} = \frac{1 - 2(2) - 4(3)}{4(2) + 2(3)}$

$$= \frac{-15}{14}$$

$$y - 3 = \frac{-15}{14}(x - 2)$$

$$y = -\frac{15}{14}x + \frac{36}{7}$$

---

29 A curve has the equation $2 \sin x + 2 \cos y = 3 \qquad 0 \leq x \leq \pi \qquad 0 \leq y \leq \pi$

(a) Find an expression for $\frac{dy}{dx}$

(b) find the coordinates of the point where $\frac{dy}{dx} = 0$

a/ $2\cos x - 2\sin y \frac{dy}{dx} = 0$

$$2\cos x = 2\sin y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{\cancel{2}\cos x}{\cancel{2}\sin y}$$

$$= \frac{\cos x}{\sin y}$$

$$\frac{\cos x}{\sin y} = 0$$

$$\cos x = 0$$
$$x = \frac{\pi}{2}$$

$$2 \sin x + 2 \cos y = 3$$

$$2 \sin\left(\frac{\pi}{2}\right) + 2 \cos y = 3$$
$$2 + 2 \cos y = 3$$
$$\cos y = \frac{1}{2}$$
$$y = \frac{1}{3}\pi$$

$$\underline{\underline{\left(\frac{\pi}{2}, \frac{\pi}{3}\right)}}$$

---

30 (a) Given that $y = 2^x$ show that $\frac{dy}{dx} = 2^x \ln 2$

(b) Find the equation to the tangent of the curve $y = 3^{(x^2)}$ at the point (2, 81)

a/ $y = 2^x$

$$\ln y = \ln 2^x$$
$$\ln y = x \ln 2$$

$$\frac{1}{y}\frac{dy}{dx} = \ln 2$$

$$\frac{dy}{dx} = y \ln 2 \qquad y = 2^x$$

$$= \underline{\underline{2^x \ln 2}}$$

b/ $y = 3^{x^2}$

$$\frac{dy}{dx} = 3^{x^2} \ln 3 \times 2x$$

when $x = 2$ $\quad \frac{dy}{dx} = 324 \ln 3$ $\quad (2, 81)$

$$y - 81 = 324 \ln 3 (x - 2)$$

31 Given that x is measured in radians, prove, from first principles, that the derivative of sin (x) is cos (x)

You may assume the formula for sin (A± B) and that as h→ 0 $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d \sin x}{dx} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \frac{(\cos h - 1)}{h} \sin x + \frac{\sin h}{h} \cos x$$

as $h \to 0$ $\quad \frac{\cos h - 1}{h} \to 0$ $\quad \frac{\sin h}{h} \to 1$

$$= 0(\sin x) + 1 \cos x$$

$$= \cos x$$

32 Given that x is measured in radians, prove, from first principles, that the derivative of cos (x) is –sin (x)

You may assume the formula for cos (A± B) and that as h→ 0 $\frac{\sin h}{h} \to 1$ and $\frac{\cos h - 1}{h} \to 0$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d \cos x}{dx} = \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos x \cosh - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h \to 0} \cos x \left(\frac{\cos h - 1}{h}\right) - \sin x \left(\frac{\sin h}{h}\right)$$

$$\boxed{\text{as } h \to 0 \quad \frac{\sin h}{h} \to 1 \quad \text{and} \quad \frac{\cos h - 1}{h} \to 0}$$

$$\frac{d \cos x}{dx} = \cos x \,(0) - \sin x \,(1)$$

$$= \underline{\underline{-\sin x}}$$

---

33 A curve has the parametric equations

$$x = 2t + 1, \qquad y = t^2 - 1$$

(a) Find the points where the curve crosses the coordinate axes.

(b) Find an expression for $\frac{dy}{dx}$ in terms of x.

a) crosses y when $x = 0$ $\quad 0 = 2t + 1$

$t = -\frac{1}{2}$

$y = \left(-\frac{1}{2}\right)^2 - 1$

$\underline{= -3/4}$

crosses x when $y = 0$ $\quad 0 = t^2 - 1$

$1 = t^2$

$t = \pm 1$

$x = 2(1) + 1 \qquad x = 2(-1) + 1$

$\underline{\underline{= 3}} \qquad \underline{\underline{= -1}}$

$\left(3, -\frac{3}{4}\right)$ and $\left(-1, -\frac{3}{4}\right)$

b/ $x = 2t + 1 \qquad y = t^2 - 1$

$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = 2t$

$$\frac{dy}{dt} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$= 2t \div 2$$

$$= t$$

$t = \frac{x-1}{2}$

$$\frac{dy}{dx} = \frac{x-1}{2}$$

---

34 A curve has the parametric equations

$$x = \tan^2 t, \qquad y = \cos t, \qquad 0 < t < \frac{\pi}{2}$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of t.

(b) Find an equation of the tangent to the curve when $t = \frac{\pi}{4}$

(c) Find a cartesian equation for the curve.

a/ $\frac{dy}{dt} = -\sin t \qquad x = (\tan t)^2$

$\frac{dx}{dt} = 2\tan t \sec^2 t$

$$\frac{dy}{dx} = \frac{-\sin t}{2\tan t \sec^2 t}$$

$$= \frac{-\sin t \cos^2 t \cot t}{2}$$

$$= -\frac{1}{2}\cos^3 t$$

b/ when $t = \frac{\pi}{4}$ $\quad \frac{dy}{dx} = -\frac{1}{2}\left(\cos\frac{\pi}{4}\right)^3$

$$= \frac{-\sqrt{2}}{8}$$

$$x = \left(\tan\left(\frac{\pi}{4}\right)\right)^2 \qquad y = \cos\left(\frac{\pi}{4}\right)$$

$$= 1 \qquad\qquad = \frac{\sqrt{2}}{2}$$

$$y - \frac{\sqrt{2}}{2} = \frac{-\sqrt{2}}{8}(x-1)$$

$$y = \frac{-\sqrt{2}}{8}x + \frac{5\sqrt{2}}{8}$$

c/ $$1 + \tan^2 t = \sec^2 t$$

$$x = \tan^2 t \quad y = \cos t$$

$$\frac{1}{y} = \sec t$$

$$\frac{1}{y^2} = \sec^2 t$$

$$1 + x = \frac{1}{y^2}$$

35 A curve has the parametric equations

$$x = \sin^2 t, \qquad y = \sin 2t, \qquad 0 < t < \pi$$

(a) Find an expression for $\frac{dy}{dx}$ in terms of t.

(b) Find an equation of the normal to the curve when t = $\frac{\pi}{6}$

(c) Find a cartesian equation for the curve.

a/

$$x = (\sin t)^2 \qquad y = \sin 2t$$

$$\frac{dx}{dt} = 2\sin t\cos t \qquad \frac{dy}{dt} = 2\cos 2t$$

$$\frac{dy}{dx} = \frac{2\cos 2t}{2\sin t\cos t}$$

$$= \frac{\cos 2t}{\sin t\cos t}$$

b/ when $t = \frac{\pi}{6}$

$$\frac{dy}{dx} = \frac{\cos(\frac{\pi}{3})}{\sin(\frac{\pi}{6})\cos(\frac{\pi}{6})}$$

$$= \frac{2\sqrt{3}}{3}$$

$\therefore$ gradient of normal $= -\frac{\sqrt{3}}{2}$

when $t = \frac{\pi}{6}$ $\quad x = (\sin(\frac{\pi}{6}))^2 \quad y = \sin(\frac{\pi}{3})$

$$= \frac{1}{4} \qquad = \sqrt{3}/2$$

$$\left(y - \frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2}\left(x - \frac{1}{4}\right)$$

$$y = -\frac{\sqrt{3}}{2}x + \frac{5\sqrt{3}}{8}$$

c/

$$y = 2\sin t\cos t$$

$$y^2 = 4\sin^2 t\cos^2 t$$

$$y^2 = 4\sin^2 t(1 - \sin^2 t)$$

$$= 4x(1 - x)$$

$$\underline{\underline{y^2 = 4x - 4x^2}}$$

36 Given that $y = \dfrac{\sin\theta}{\sin\theta + \cos\theta}$

Show that $\dfrac{dy}{d\theta} = \dfrac{1}{\sin 2\theta + 1}$

$u = \sin\theta$ $\qquad$ $v = \sin\theta + \cos\theta$

$\dfrac{du}{d\theta} = \cos\theta$ $\qquad$ $\dfrac{dv}{d\theta} = \cos\theta - \sin\theta$

$$\frac{dy}{d\theta} = \frac{\cos\theta(\sin\theta + \cos\theta) - \sin\theta(\cos\theta - \sin\theta)}{(\sin\theta + \cos\theta)^2}$$

$$= \frac{\cos\theta\sin\theta + \cos^2\theta - \cos\theta\sin\theta + \sin^2\theta}{(\sin\theta + \cos\theta)^2}$$

$$= \frac{1}{(\sin\theta + \cos\theta)^2}$$

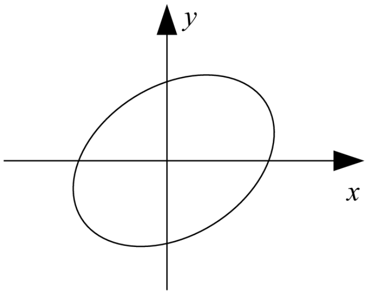
$$= \frac{1}{\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta}$$

$$= \frac{1}{1 + 2\sin\theta\cos\theta}$$

$$= \frac{1}{1 + \sin 2\theta}$$

$$= \frac{1}{\sin 2\theta + 1}$$

37 The diagram show a sketch of the curve with equation $x^2 - xy + 2y^2 = 20$



(a) Show that $\dfrac{dy}{dx} = \dfrac{2x - y}{x - 4y}$

(b) Find the points on the curve where $\dfrac{dy}{dx} = 0$

a/

$$x^2 + xy + 2y^2 = 20$$

$$u = -x \qquad v = y$$

$$\frac{du}{dx} = -1 \qquad \frac{dv}{dx} = \frac{dy}{dx}$$

$$2x - y - x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0$$

$$2x - y = x\frac{dy}{dx} - 4y\frac{dy}{dx}$$

$$2x - y = \frac{dy}{dx}(x - 4y)$$

$$\frac{dy}{dx} = \frac{2x - y}{x - 4y}$$

b/

$$\frac{2x - y}{x - 4y} = 0$$

$$2x - y = 0$$

$$2x = y \qquad x^2 - xy + 2y^2 = 20$$

$$x^2 - x(2x) + 2(2x)^2 = 20$$

$$x^2 - 2x^2 + 8x^2 = 20$$

$$7x^2 = 20$$

$$x^2 = \frac{20}{7}$$

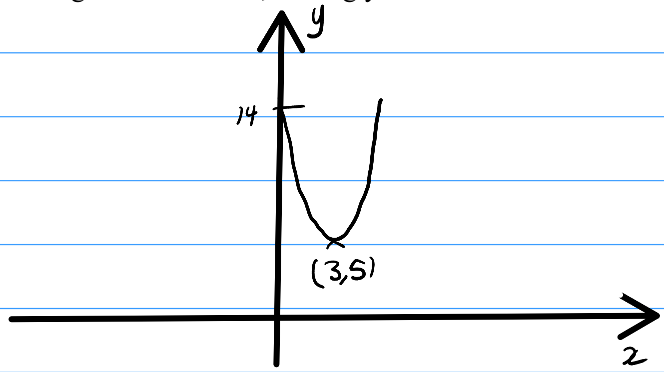
$$x = \pm \frac{2\sqrt{35}}{7}$$

$$\left(\frac{2\sqrt{35}}{7}, \frac{\sqrt{35}}{7}\right) \text{ and } \left(\frac{-2\sqrt{35}}{7}, -\frac{\sqrt{35}}{7}\right)$$

38 (a) Sketch the graph of $y = (3 - x)^2 + 5, \quad 0 \leq x \leq 6$

The line with equation $x + y = k$, where k is a constant, intersects the curve at two distinct points.

(b) State the range of values of k, writing your answer in set notation.



b) $k \leqslant 14$

$x + y = k$ is a tangent when $\frac{dy}{dx} = -1$

$$\frac{dy}{dx} = -2(3 - x)$$

$$-2(3 - x) = -1$$
$$2(3 - x) = 1$$
$$6 - 2x = 1$$
$$2x = 5$$
$$x = 2.5 \qquad \therefore k > 2.5$$

$$2.5 < k \leq 14$$

$$\{k : 2.5 < k \leqslant 14\}$$

39

$$\frac{10x+3-4x^2}{(1-x)(2x+1)} = A + \frac{B}{1-x} + \frac{C}{2x+1} \qquad x > 1$$

(a) Find the values of A, B and C.

$$f(x) = \frac{10x+3-4x^2}{(1-x)(2x+1)}$$

(b) Prove that f(x) is an increasing function.

a/ $10x + 3 - 4x^2 = A(1-x)(2x+1) + B(2x+1) + C(1-x)$

Let $x = 1$ $\quad 9 = 3B$

$B = 3$

Let $x = -\frac{1}{2}$ $\quad -3 = \frac{3}{2}C$

$C = -2$

Let $x = 0$ $\quad 3 = A + 3 - 2$

$A = 2$

$$2 + \frac{3}{1-x} - \frac{2}{2x+1}$$

b/ $f(x) = 2 + 3(1-x)^{-1} - 2(2x+1)^{-1}$

$f'(x) = 3(1-x)^{-2} + 4(2x+1)^{-2}$

$$= \frac{3}{(1-x)^2} + \frac{4}{(2x+1)^2}$$

$(1-x)^2$ and $(2x+1)^2$ will always be positive and positive ÷ positive = positive

$\therefore$ $f'(x)$ is positive

and $f(x)$ is an increasing function.

40 The temperature, K°C, of water in a kettle is modelled by the formula $K = 20 + 75e^{-0.2t}$
Where t is the time in minutes since measurement began.

(a) State the starting temperature of the kettle.

(b) Find $\frac{dK}{dt}$

a/ when $t = 0$ $\quad K = 20 + 75$

$$= 95°C$$

b/ $$\frac{dK}{dt} = -15e^{-0.2t}$$

41

$$y = \frac{x^2 + 4x}{(x+2)^2} \qquad x \neq -2$$

(a) Show that $\frac{dy}{dx} = \frac{A}{(x+2)^n}$ where A and n are constants to be found.

(b) Hence deduce the range of values for x for which $\frac{dy}{dx} < 0$

a/ $u = x^2 + 4x \qquad v = (x+2)^2$

$\frac{du}{dx} = 2x + 4 \qquad \frac{dv}{dx} = 2(x+2)$

$$\frac{dy}{dx} = \frac{(2x+4)(x+2)^2 - 2(x+2)(x^2+4x)}{(x+2)^4}$$

$$= \frac{(2x+4)(x^2+4x+4) - 2(x^3+4x^2+2x^2+8x)}{(x+2)^4}$$

$$= \frac{2x^3+8x^2+8x+4x^2+16x+16-2(x^3+6x^2+8x)}{(x+2)^4}$$

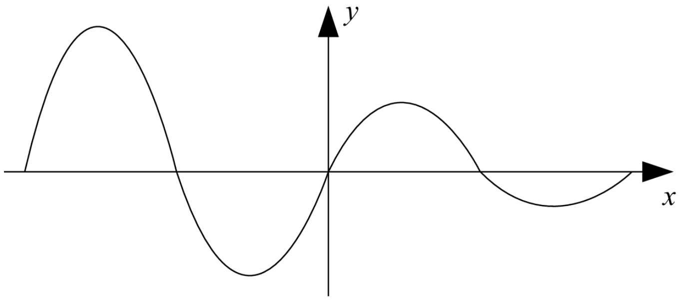
$$= \frac{2x^3+12x^2+24x+16-2x^3-12x^2-16x}{(x+2)^4}$$

$$= \frac{8x+16}{(x+2)^4}$$

$$= \frac{8(x+2)}{(x+2)^4}$$

$$= \frac{8}{(x+2)^3}$$

42 The curve C with equation $f(x) = \dfrac{\sin 2x}{e^x}$ $-\pi < x < \pi$ is shown in the diagram.



(a) Show that the x-coordinates of the turning points of C satisfy the equation $\tan 2x = 2$

(b) Hence find the coordinates of the turning points

a/

$u = \sin 2x$ $\qquad v = e^x$

$\dfrac{du}{dx} = 2\cos 2x$ $\qquad \dfrac{dv}{dx} = e^x$

$$f'(x) = \frac{2e^x \cos 2x - e^x \sin 2x}{e^{2x}}$$

Turning points are where $f'(x) = 0$

$$\frac{2e^2 \cos 2x - e^x \sin 2x}{e^{2z}} = 0$$

$$e^x(2\cos 2x - \sin 2x) = 0$$

cannot equal zero

$$2\cos 2x - \sin 2x = 0$$

$$2\cos 2x = \sin 2x$$

$$\underline{\underline{2 = \tan 2x}}$$

b/

$\tan 2x = 2$

$2x = 1.107,\ 4.249,\ -2.034,\ -5.176$

$x = \underline{-2.59},\ \underline{-1.02},\ \underline{0.554},\ \underline{2.12}$

$\underline{\underline{(-2.59, 11.9)}}$ $\underline{\underline{(-1.02, -2.47)}}$ $\underline{\underline{(0.554, 0.514)}}$ $\underline{\underline{(2.12, -0.107)}}$

43 The curve C has the equation $x = 2\tan 2y$ $\quad \frac{-\pi}{4} < y < \frac{\pi}{4}$

Show that, for all points (x, y) lying on C,

$1 + \tan^2 2y = \sec^2 2y$

$$\frac{dy}{dx} = \frac{a}{x^2 + b}$$

where a and b are constants to be found

$$\frac{dx}{dy} = 4\sec^2 2y$$

$$\frac{dy}{dx} = \frac{1}{4\sec^2 2y}$$

$$= \frac{1}{4(1+\tan^2 2y)}$$

$x = 2\tan 2y$

$x^2 = 4\tan^2 2y$

$$= \frac{1}{4 + 4\tan^2 2y}$$

$$= \frac{1}{4 + x^2}$$

---

44 The curve C has the equation $y = x^x$ $\quad x > 0$

Find, by firstly taking logarithms, the x-coordinate of the turning point of C.

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

$u = x \qquad v = \ln x$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \frac{1}{x}$

$$\frac{1}{y}\frac{dy}{dx} = 1 + \ln x$$

Turning point where $\frac{dy}{dx} = 0$

$$0 = 1 + \ln x$$

$$-1 = \ln x$$

$$x = e^{-1}$$

45 A storage tank is modelled in the shape of a hollow cylinder.

The walls of the tank are assumed to have negligible thickness.

The cylinder has radius r metres and height h metres.

The volume of the tank is 12 m^3

(a) Show that the surface area of the tank, in m^2, is $2\pi r^2 + \frac{24}{r}$

(b) Use calculus to find the radius of the tank for which the surface area is a minimum.

(c) Calculate the minimum surface area of the tank, giving your answer to the nearest integer.

a/ volume $= \pi r^2 h$

$\pi r^2 h = 12$

$h = \frac{12}{\pi r^2}$

$s.a = 2\pi r^2 + 2\pi r h$

$= 2\pi r^2 + 2\cancel{\pi r}\left(\frac{12}{\cancel{\pi} r^{\cancel{2}}}\right)$

$= 2\pi r^2 + \frac{24}{r}$

b/ $\frac{ds}{dr} = 4\pi r - 24r^{-2}$

min where $\frac{ds}{dr} = 0$ $\quad 4\pi r - \frac{24}{r^2} = 0$

$4\pi r^3 - 24 = 0$

$r^3 = \frac{24}{4\pi}$

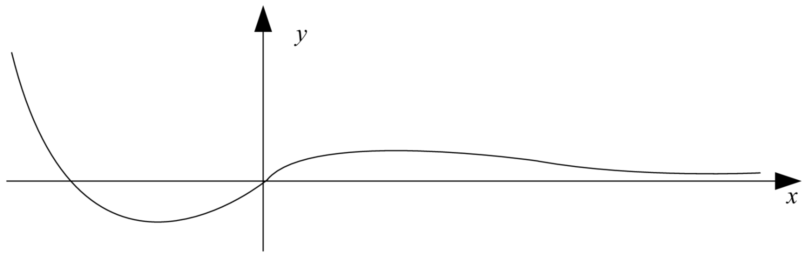
$r = \sqrt[3]{\frac{24}{4\pi}}$

$= \underline{\underline{1.24 \text{ m}}}$

c/ $s.a = 2\pi(1.24)^2 + \frac{24}{1.24}$

$= \underline{\underline{29 \text{ m}^2}}$

46 The diagram shows the graph with equation $y = f(x)$ where $f(x) = (2x^2 + 3x)e^{-x}$ $x \in \mathbb{R}$



(a) Show that $f'(x) = e^{-x}(x + 3 - 2x^2)$

(b) Hence, find, in simplest form, the exact coordinates of the stationary points of C

The function g and the function h are defined by

$$g(x) = 2f(x) \qquad x \in \mathbb{R}$$

$$h(x) = f(x) + 2 \qquad x > 0$$

(c) Find (i) the range of g
(ii) the range of h

$u = 2x^2 + 3x$ $\quad v = e^{-x}$

$\frac{du}{dx} = 4x + 3$ $\quad \frac{dv}{dx} = -e^{-x}$

$f'(x) = e^{-x}(4x+3) - e^{-x}(2x^2 + 3x)$

$f'(x) = e^{-x}(4x + 3 - (2x^2 + 3x))$

$= e^{-x}(4x + 3 - 2x^2 - 3x)$

$= e^{-x}(x + 3 - 2x^2)$

b/ stationary points where $f'(x) = 0$

$e^{-x}(x + 3 - 2x^2) = 0$

$x + 3 - 2x^2 = 0$

$2x^2 - x - 3 = 0$

$(2x - 3)(x + 1) = 0$

$x = \frac{3}{2}$ $\quad x = -1$

$\underline{\underline{\left(\frac{3}{2}, 9e^{-3/2}\right)}}$ and $\underline{\underline{(-1, -e)}}$

c) Min point at $(-1, -e)$ $\quad f(x) \geqslant -e$

$g(x) \geqslant -2e$

ii) max point at $\left(\frac{3}{2}, 9e^{-\frac{3}{2}}\right)$

$0 < f(x) \leqslant 9e^{-\frac{3}{2}}$ for $x > 0$

$2 < h(x) \leqslant 9e^{-\frac{3}{2}} + 2$

---

47 Curve C is given by $x + 5 = \sin x \cos x + 2y \qquad 0 < x < \pi$

(a) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \sin^2 x$

(b) Find the coordinates of the point of inflection of C.

$u = \sin x \qquad v = \cos x$

$\frac{du}{dx} = \cos x \qquad \frac{dv}{dx} = -\sin x$

$1 = \cos^2 x - \sin^2 x + 2\frac{dy}{dx}$

$1 = (1 - \sin^2 x) - \sin^2 x + 2\frac{dy}{dx}$

$1 = 1 - 2\sin^2 x + 2\frac{dy}{dx}$

$0 = -2\sin^2 x + 2\frac{dy}{dx}$

$2\sin^2 x = 2\frac{dy}{dx}$

$\frac{dy}{dx} = \sin^2 x$

b) $\frac{d^2y}{dx^2} = 2\sin x\cos x$

$= \sin 2x$

point of inflection where $\frac{d^2y}{dx^2} = 0$

$\sin 2x = 0$

$2x = \cancel{0}, \pi$

$x = \frac{\pi}{2}$

$\sin(3.1) = 0.04$ positive

$\sin(3.2) = -0.06$ negative

$\therefore$ point of inflection

$\frac{\pi}{2} + 5 = \sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) + 2y$

$\frac{\pi}{2} + 5 = 2y$

$y = \frac{\pi}{4} + \frac{5}{2}$

$= \frac{\pi + 10}{4}$

$\left(\frac{\pi}{2}, \frac{\pi+10}{4}\right)$

---

48 A function f is given by $f(x) = \dfrac{5 + \ln x}{3 + 2\ln x}$ $x > 0$

Prove that f is a decreasing function.

$u = 5 + \ln x$ $v = 3 + 2\ln x$

$\frac{du}{dx} = \frac{1}{x}$ $\frac{dv}{dx} = \frac{2}{x}$

$$f'(x) = \frac{\frac{1}{x}(3 + 2\ln x) - \frac{2}{x}(5 + \ln x)}{(3 + 2\ln x)^2}$$

$$= \frac{1}{x}\frac{(3 + 2\ln x - 2(5 + \ln x))}{(3 + 2\ln x)^2}$$

$$= \frac{\frac{1}{x}\left(3 + 2\ln x - 10 - 2\ln x\right)}{(3 + 2\ln x)^2}$$

$$= \frac{1}{x}\left(\frac{-7}{(3+2\ln x)^2}\right)$$

for $x > 0$ $\quad \frac{1}{x} > 0$ and $(3 + 2\ln x)^2 > 0$

positive $\times$ $\frac{\text{negative}}{\text{positive}}$ = negative

$\therefore$ $f(x)$ is a decreasing function.

---

49 A curve C has the equation $y = \frac{x^2+5}{x} + \ln x \qquad x > 0$

(a) Show that $\frac{dy}{dx} = \frac{x^2+x-5}{x^2}$

(b) Hence find the coordinates of the turning point of C.

a/ $$y = x + 5x^{-1} + \ln x$$

$$\frac{dy}{dx} = 1 - 5x^{-2} + \frac{1}{x}$$

$$= 1 - \frac{5}{x^2} + \frac{1}{x}$$

$$= \frac{x^2}{x^2} - \frac{5}{x^2} + \frac{x}{x^2}$$

$$= \frac{x^2 + x - 5}{x^2}$$

b/ $$\frac{x^2+x-5}{x^2} = 0$$

$x^2 + x - 5 = 0$

$x = 1.79$ (x cannot equal -2.79 as $x > 0$)

$(1.79, 5.17)$

$$y = a^x$$
$$\frac{dy}{dx} = a^x \ln a$$

50 A curve is defined by the parametric equations

$$x = 3 \times 2^{-t} + 2$$
$$y = 5 \times 2^{t} - 3$$

(a) Show that $\frac{dy}{dx} = -\frac{5}{3} \times 2^{2t}$

(b) Find a Cartesian equation for the curve.

a/ $\frac{dx}{dt} = -3 \cdot 2^{-t} \ln 2$ $\qquad \frac{dy}{dt} = 5 \cdot 2^{t} \ln 2$

$$\frac{dy}{dx} = \frac{5 \cdot 2^{t} \cancel{\ln 2}}{-3 \cdot 2^{-t} \cancel{\ln 2}}$$

$$\frac{dy}{dx} = -\frac{5}{3} \times \frac{2^{t}}{2^{-t}}$$

$$= -\frac{5}{3} \times 2^{2t}$$

b/ $x = \frac{3}{2^t} + 2$

$$x \cdot 2^t = 3 + 2 \times 2^t$$
$$x \cdot 2^t - 2 \cdot 2^t = 3$$
$$2^t(x-2) = 3$$
$$2^t = \frac{3}{x-2}$$
$$y = 5 \times 2^t - 3$$
$$y = 5\left(\frac{3}{x-2}\right) - 3$$
$$y = \frac{15}{x-2} - 3$$

51 A six-sided box, in the shape of a cuboid, is made from a sheet metal. The base of the box is x cm by y cm and the height of the box is x cm.

The volume of the box is 5000 cm^2.

(a) Show that the the area of sheet metal, A cm^2, is given by $A = \frac{20000}{x} + 2x^2$ **(4)**

(b) Use calculus to show to find the value of x for which A is stationary. **(4)**

(c) Prove that this value of x gives a minimum value of A. **(2)**

(d) Calculate the minimum area of sheet metal needed to make the box. **(2)**

$v = x^2 y$

$x^2 y = 5000$

$y = \frac{5000}{x^2}$

$a = 4xy + 2x^2$

$a = 4x\left(\frac{5000}{x^2}\right) + 2x^2$

$= \frac{20000}{x} + 2x^2$

b/ $\frac{dA}{dx} = -20000x^{-2} + 4x$

stationary when $\frac{dA}{dx} = 0$

$-\frac{20000}{x^2} + 4x = 0$

$-20000 + 4x^3 = 0$

$4x^3 = 20000$

$x^3 = 5000$

$x = 36.8$ cm

c/ $\frac{d^2A}{dx^2} = 40000x^{-3} + 4$

when $x = 36.8$ $\frac{d^2A}{dx^2} = 1090$ positive $\therefore$ minimum

d/ $A = \frac{20000}{x} + 2x^2$

when $x = 36.8$ $A = 3257$ cm^2

52 Find the coordinates of the stationary point of the curve with equation

$$x^2 + 2x = e^y - 3$$

$$2x + 2 = e^y \frac{dy}{dx}$$

stationary point where $\frac{dy}{dx} = 0$

$$2x + 2 = 0$$
$$x = -1$$

when $x = -1$

$$1 - 2 = e^y - 3$$
$$2 = e^y$$
$$y = \ln 2$$

$$\underline{\underline{(-1, \ln 2)}}$$

53 A function f is defined by $f(x) = \dfrac{x}{\sqrt{2x-3}}$

(a) State the maximum possible domain of f.

(b) Use the quotient rule to show that $f'(x) = \dfrac{x-3}{(2x-3)^{\frac{3}{2}}}$

(c) Show that the graph of $y = f(x)$ has exactly one point of inflection.

a/ $$x > \frac{3}{2}$$

b/ $u = x$ $\qquad v = (2x-3)^{\frac{1}{2}}$

$\dfrac{du}{dx} = 1$ $\qquad \dfrac{dv}{dx} = (2x-3)^{-\frac{1}{2}}$

$$f'(x) = \frac{(2x-3)^{\frac{1}{2}} - x(2x-3)^{-\frac{1}{2}}}{(2x-3)} \qquad \begin{matrix} \times (2x-3)^{\frac{1}{2}} \\ \times (2x-3)^{\frac{1}{2}} \end{matrix}$$

$$= \frac{2x-3-x}{(2x-3)^{\frac{3}{2}}}$$

$$= \frac{x-3}{(2x-3)^{3/2}}$$

c/ $u = x-3$ $\qquad v = (2x-3)^{3/2}$

$\dfrac{du}{dx} = 1$ $\qquad \dfrac{dv}{dx} = 3(2x-3)^{\frac{1}{2}}$

$$f''(x) = \frac{(2x-3)^{3/2} - 3(x-3)(2x-3)^{\frac{1}{2}}}{(2x-3)^3}$$

point of inflection where $f''(x) = 0$

$$(2x-3)^{3/2} - 3(x-3)(2x-3)^{\frac{1}{2}} = 0$$

$$(2x-3)^{\frac{1}{2}}(2x-3-3x+9) = 0$$

$$(2x-3)^{\frac{1}{2}}(6-x) = 0$$

$$(2x-3)^{\frac{1}{2}} = 0$$

$x = \frac{3}{2}$ $\quad$ $x = 6$

$\times$

x must be greater than $\frac{3}{2}$ $\therefore$ $\underline{x = 6}$

$f''(5.9) > 0$

$f''(6.1) < 0$ $\therefore$ point of inflection

54 Given $y = e^{kx}$, where k is a constant, find $\frac{dy}{dx}$ in terms of k.

$y = e^{kx}$

$$\frac{dy}{dx} = ke^{kx}$$

55 The volume of a sphere is increasing at a constant rate of 2 cm^3s^{-1}

Show that the rate of increase of the radius when $r = 2$ cm is $\frac{a}{\pi}$, where a is a constant.

Volume of a sphere = $\frac{4}{3}\pi r^3$

$\frac{dV}{dt} = 2$ $\qquad$ $\frac{dV}{dr} = 4\pi r^2$

$$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$$

$$= \frac{1}{4\pi r^2} \times 2$$

$$= \frac{1}{2\pi r^2}$$

when $r = 2$ $\qquad$ $\frac{dr}{dt} = \frac{1}{8\pi}$

$$a = \frac{1}{8}$$

56 Given $y = 3^x$, show that $\dfrac{dy}{dx} = 3^x \ln 3$

$$y = 3^x$$

$$\ln y = x \ln 3$$

$$\frac{1}{y}\frac{dy}{dx} = \ln 3$$

$$\frac{dy}{dx} = y \ln 3 \qquad y = 3^x$$

$$= \underline{\underline{3^x \ln 3}}$$

---

57 A curve, C, has the equation $y = \dfrac{e^{2x+1}}{x^2}$

Show that C has exactly one stationary point.

Fully justify your answer.

$$u = e^{2x+1} \qquad v = x^2$$

$$\frac{du}{dx} = 2e^{2x+1} \qquad \frac{dv}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x^2 e^{2x+1} - 2x e^{2x+1}}{x^4}$$

stationary point where $\dfrac{dy}{dx} = 0$

$$2x^2 e^{2x+1} - 2x e^{2x+1} = 0$$

$$e^{2x+1}(2x^2 - 2x) = 0$$

Cannot $= 0$

$$2x^2 - 2x = 0$$

$$2x(x-1) = 0$$

$$x = 0 \qquad x = 1$$

when $x = 0$ C is undefined

$\therefore$ $\underline{\underline{\text{one stationary point when } x = 1}}$

58 A curve, C, has the equation $xy^2 + x^2y = 10$

(a) Prove that the curve does not intersect the coordinate axes.

(b) Show that $\frac{dy}{dx} = -\frac{y(2x+y)}{x(x+2y)}$

(c) Find the exact coordinates of the stationary point.

a/ crosses x if there is an intersection with $y=0$

$$x(0) + x^2(0) = 10$$

$0 \neq 10$ $\therefore$ no sol.

crosses y if intersection with $x=0$

$$(0)y^2 + (0)y = 10$$

$0 \neq 10$ $\therefore$ no intersection

b/ $u = x$ $\quad v = y^2$ $\qquad u = x^2$ $\quad v = y$

$\frac{du}{dx} = 1$ $\quad \frac{dv}{dx} = 2y\frac{dy}{dx}$ $\qquad \frac{du}{dx} = 2x$ $\quad \frac{dv}{dx} = \frac{dy}{dx}$

$$y^2 + 2xy\frac{dy}{dx} + x^2\frac{dy}{dx} + 2xy = 0$$

$$\frac{dy}{dx}(x^2 + 2xy) = -y^2 - 2xy$$

$$= -\frac{(y^2 + 2xy)}{x^2 + 2xy}$$

$$= -\frac{y(y+2x)}{x(x+2y)}$$

c/ $$\frac{-y(y+2x)}{x(x+2y)} = 0$$

$$-y(y+2x) = 0$$

$y = 0$ $\quad y + 2x = 0$

$\times$

$$xy^2 + x^2y = 10 \qquad y = -2x$$

$$x(-2x)^2 + x^2(-2x) = 10$$
$$4x^3 - 2x^3 = 10$$
$$2x^3 = 10$$
$$x^3 = 5$$
$$x = \sqrt[3]{5}$$
$$y = -2\sqrt[3]{5}$$

$$\underline{\underline{\left(\sqrt[3]{5},\ -2\sqrt[3]{5}\right)}}$$

---

59 A curve, C, has the equation $x^2y + xy = 6$

When $x > 0$, find the equation of the tangent to C when $y = 1$.

when $y = 1 \quad x^2 + x = 6$
$$x^2 + x - 6 = 0$$
$$(x+3)(x-2) = 0$$
$$x = -3 \quad x = 2$$
$$x > 0 \quad \therefore \quad x = 2$$

$$u = x^2 \quad v = y \qquad\qquad u = x \quad v = y$$
$$\frac{du}{dx} = 2x \quad \frac{dv}{dx} = \frac{dy}{dx} \qquad\qquad \frac{du}{dx} = 1 \quad \frac{dv}{dx} = \frac{dy}{dx}$$

$$x^2\frac{dy}{dx} + 2xy + x\frac{dy}{dx} + y = 0$$

at $(2,1)$ $\quad 4\frac{dy}{dx} + 4 + 2\frac{dy}{dx} + 1 = 0$

$$6\frac{dy}{dx} = -5$$
$$\frac{dy}{dx} = -\frac{5}{6}$$

$$y - 1 = -\frac{5}{6}(x - 2)$$
$$y = -\frac{5}{6}x + \frac{8}{3}$$

60 A curve, C, has the equation $5\sin y + x\cos y = Ax$

where A is a constant.

C passes through the point $\left(\sqrt{3}, \frac{\pi}{3}\right)$

(a) Show that $A = 3$

(b) Show that $\frac{dy}{dx} = \frac{3 - \cos y}{5\cos y - x\sin y}$

(c) Hence find the equation of the tangent to C at P.

a/

$$5\sin\frac{\pi}{3} + \sqrt{3}\cos\frac{\pi}{3} = \sqrt{3}A$$

$$\frac{5\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \sqrt{3}A$$

$$3\sqrt{3} = A\sqrt{3}$$

$$\underline{\underline{A = 3}}$$

b/

$$5\sin y + x\cos y = 3x$$

$u = x$ $\quad v = \cos y$

$\frac{du}{dx} = 1$ $\quad \frac{dv}{dx} = -\sin y\frac{dy}{dx}$

$$5\cos y\frac{dy}{dx} + \cos y - x\sin y\frac{dy}{dx} = 3$$

$$\frac{dy}{dx}(5\cos y - x\sin y) = 3 - \cos y$$

$$\frac{dy}{dx} = \frac{3 - \cos y}{5\cos y - x\sin y}$$

c/ at $\left(\sqrt{3}, \frac{\pi}{3}\right)$

$$\frac{dy}{dx} = \frac{3 - \frac{1}{2}}{5\left(\frac{1}{2}\right) - \left(\frac{3}{2}\right)}$$

$$= \frac{5}{2}$$

$$\underline{\underline{y - \frac{\pi}{3} = \frac{5}{2}\left(x - \sqrt{3}\right)}}$$

61 A curve has parametric equations $x = 3t + \frac{1}{t}$ and $y = 3t - \frac{1}{t}$ for $t \neq 0$

(a) Find $\frac{dy}{dx}$ in terms of t, giving your answer in its simplest form. **(4)**

(b) Explain why the curve has no stationary points. **(2)**

(c) By considering $x + y$, or otherwise, find a cartesian equation of the curve, giving your answer in a form not involving fractions or brackets. **(4)**

$$\frac{dx}{dt} = 3 - t^{-2} \qquad \frac{dy}{dt} = 3 + t^{-2}$$

$$\frac{dy}{dx} = \frac{3 + t^{-2}}{3 - t^{-2}} \qquad \frac{\times t^2}{\times t^2}$$

$$= \frac{3t^2 + 1}{3t^2 - 1}$$

b/ stationary points where $\frac{3t^2 + 1}{3t^2 - 1} = 0$

$$3t^2 + 1 = 0$$
$$3t^2 = -1$$
$$t^2 = -\tfrac{1}{3}$$

t^2 is always positive $\therefore$ cannot equal $-\frac{1}{3}$

$\therefore$ no sols/ no stationary points.

c/ $$x + y = 3t + \frac{1}{t} + 3t - \frac{1}{t}$$

$$x + y = 6t$$
$$t = \frac{1}{6}(x + y)$$

$$x = 3t + \frac{1}{t}$$

$$x = 3\left(\frac{1}{6}\right)(x+y) + \frac{1}{\frac{1}{6}(x+y)} \qquad \times (x+y)$$

$$x(x+y) = \frac{1}{2}(x+y)^2 + 6 \qquad \times 2$$
$$2x(x+y) = (x+y)^2 + 12$$
$$2x^2 + \cancel{2xy} = x^2 + \cancel{2xy} + y^2 + 12$$
$$\underline{x^2 = y^2 + 12}$$

62 A curve has the equation $y = x \ln x$

(a) Find the coordinates of the turning point of the curve.

(b) Determine whether this turning point is a maximum or a minimum.

a/ $u = x \qquad v = \ln x$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = 1 + \ln x$$

$$1 + \ln x = 0$$
$$\ln x = -1$$
$$x = e^{-1}$$
$$y = e^{-1} \ln e^{-1}$$
$$= -e^{-1}$$

$$(e^{-1}, -e^{-1})$$

b/ $$\frac{d^2y}{dx^2} = \frac{1}{x}$$

when $x = e^{-1}$ $\frac{d^2y}{dx^2} = e$ positive $\therefore$ minimum

63 A curve has the equation $x^3 - x^2y - 2y - 5 = 0$

(a) Show that $\frac{dy}{dx} = \frac{3x^2 - 2xy}{x^2 + 2}$

(b) Find the equation of the normal to the curve at the point (1, 2).

a/

$u = -x^2 \quad v = y$

$\frac{du}{dx} = -2x \quad \frac{dv}{dx} = \frac{dy}{dx}$

$$3x^2 - x^2\frac{dy}{dx} - 2xy - 2\frac{dy}{dx} = 0$$

$$3x^2 - 2xy = x^2\frac{dy}{dx} + 2\frac{dy}{dx}$$

$$3x^2 - 2xy = \frac{dy}{dx}(x^2 + 2)$$

$$\underline{\underline{\frac{dy}{dx} = \frac{3x^2 - 2xy}{x^2 + 2}}}$$

b/ at $(1, 2)$ $\quad \frac{dy}{dx} = -\frac{1}{3}$

$\therefore$ gradient of normal $= 3$

$$(y - 2) = 3(x - 1)$$

$$\underline{\underline{y = 3x - 1}}$$

64 Let f(x) = $x^3 + 5x$. Use differentiation from first principles to show that $f'(x) = 3x^2 + 5$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^3 + 5(x+h) - (x^3 + 5x)}{h}$$

$$= \lim_{h \to 0} \frac{(x^2 + 2xh + h^2)(x+h) + 5x + 5h - x^3 - 5x}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + x^2h + 2x^2h + 2xh^2 + xh^2 + h^3 + \cancel{5x} + 5h - \cancel{x^3} - \cancel{5x}}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 + 5h}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 + 5$$

as $h \to 0$ $3xh \to 0$ and $h^2 \to 0$

$$f'(x) = \underline{\underline{3x^2 + 5}}$$

65 A curve has the equation $y = a^{x^2}$, where a is a constant greater than 1.

(a) Show that $\frac{dy}{dx} = 2xa^{x^2}\ln a$

(b) The tangent at the point $(1, a)$ passes through the point $\left(\frac{1}{2}, 0\right)$.

Find the value of a, giving your answer in an exact form.

(c) By considering $\frac{d^2y}{dx^2}$ show that the curve is convex for all values of x.

a/ $\ln y = \ln a^{x^2}$

$\ln y = x^2(\ln a)$

$\frac{1}{y}\frac{dy}{dx} = 2x(\ln a)$

$\frac{dy}{dx} = 2xy\ln a$ $\qquad y = a^{x^2}$

$\frac{dy}{dx} = 2xa^{x^2}\ln a$

b/ at $(1, a)$ $\qquad \frac{dy}{dx} = 2a\ln a$

passes through $\left(\frac{1}{2}, 0\right)$ and $(1, a)$

$m = \frac{a}{1/2}$

$= 2a$

$2a = 2a\ln a$

$0 = 2a - 2a\ln a$

$0 = 2a(1 - \ln a)$

a cannot be zero $\qquad \therefore \ln a = 1$

$a = e$

c/ $\frac{dy}{dx} = 2xa^{x^2}\ln a$

$= 2xe^{x^2}$

$u = 2x \qquad v = e^{x^2}$

$\frac{du}{dx} = 2 \qquad \frac{dv}{dx} = 2xe^{x^2}$

$$\frac{d^2y}{dx^2} = 4x^2e^{x^2} + 2e^{x^2}$$

$$= e^{x^2}(4x^2 + 2)$$

e^{x^2} is always positive $\quad$ $4x^2 + 2$ is always positive

positive × positive = positive

$\therefore \frac{d^2y}{dx^2}$ is always positive so convex

---

66 Differentiate with respect to x:

(a) $(2x + 5)^5$

(b) $3x^2 \ln x$

a/ $(2x+5)^5$

$10(2x+5)^4$

b/ $u = 3x^2 \qquad v = \ln x$

$\frac{du}{dx} = 6x \qquad \frac{dv}{dx} = \frac{1}{x}$

$6x \ln x + 3x$

67 A solid right circular cylinder has radius r cm and height h cm.

The total surface area of the cylinder is 800 cm^2 .

(a) Show that the volume, V cm^3 , of the cylinder is given by $V = 400r - \pi r^3$

Given that r varies,

(b) use calculus to find the maximum value of V, to the nearest cm^3.

(c) Justify that the value of V you have found is a maximum.

$v = \pi r^2 h$

$$s = 2\pi r^2 + 2\pi r h$$
$$800 = 2\pi r^2 + 2\pi r h$$
$$400 = \pi r^2 + \pi r h$$
$$\frac{400 - \pi r^2}{\pi r} = h$$

$$v = \cancel{\pi} r^{\cancel{2}} \left(\frac{400 - \pi r^2}{\cancel{\pi r}} \right)$$
$$= r(400 - \pi r^2)$$
$$= \underline{\underline{400r - \pi r^3}}$$

b/ $$\frac{dv}{dr} = 400 - 3\pi r^2$$

max where $\frac{dv}{dr} = 0$

$$400 - 3\pi r^2 = 0$$
$$400 = 3\pi r^2$$
$$r^2 = \frac{400}{3\pi}$$
$$r = \sqrt{\frac{400}{3\pi}} = 6.51$$

$$v = 400(6.51) - \pi(6.51)^3$$
$$= \underline{\underline{1737 \text{ cm}^3}}$$

c/ $$\frac{d^2v}{dr^2} = -6\pi r$$

$$= -6\pi(6.51) \qquad \frac{d^2v}{dr^2} < 0 \quad \therefore \quad \underline{\underline{\text{maximum}}}$$

68 A curve has the equation $y = x^3 + kx^2 + 5$, where k is a constant.

Given that the curve has a point of inflection where $x = 2$, show that $k = -6$.

$$\frac{dy}{dx} = 3x^2 + 2kx$$

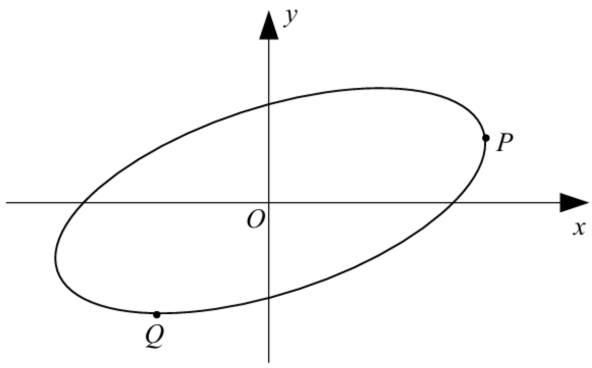
$$\frac{d^2y}{dx^2} = 6x + 2k$$

when $x = 2$ $\quad \frac{d^2y}{dx^2} = 0$ $\quad 6(2) + 2k = 0$

$12 + 2k = 0$

$k = -6$

69



The diagram shows the curve with equation $x^2 - 2xy + 4y^2 - 12 = 0$

(a) Show that $\frac{dy}{dx} = \frac{x - y}{x - 4y}$ **(3)**

At the point P on the curve the tangent to the curve is parallel to the y-axis and at the point Q on the curve the tangent to the curve is parallel to the x-axis.

(b) Show that the exact distance PQ, is $k\sqrt{5}$ where k is a rational number to be determined. **(7)**

$u = -2x \quad v = y$

$2x - 2y - 2x\frac{dy}{dx} + 8y\frac{dy}{dx} = 0 \qquad \frac{du}{dx} = -2 \quad \frac{dv}{dx} = \frac{dy}{dx}$

$$2x - 2y = 2x\frac{dy}{dx} - 8y\frac{dy}{dx}$$

$$x - y = x\frac{dy}{dx} - 4y\frac{dy}{dx}$$

$$x - y = \frac{dy}{dx}(x - 4y)$$

$$\frac{dy}{dx} = \frac{x - y}{x - 4y}$$

at Q $\frac{dy}{dx} = 0$ $\quad \frac{x-y}{x-4y} = 0$

$$x - y = 0$$
$$x = y$$

$$x^2 - 2xy + 4y^2 - 12 = 0$$

$$x^2 - 2x^2 + 4x^2 - 12 = 0 \qquad \text{sub } y=x \text{ in}$$
$$3x^2 - 12 = 0$$
$$x^2 = 4$$
$$x = \pm 2$$

at Q $x < 0$ $\therefore$ $\underline{\underline{x = -2}}$ $\quad \underline{\underline{y = -2}}$ $\quad (-2, -2)$

at P $\frac{dy}{dx}$ is undefined $\therefore$ $x - 4y = 0$

$$x = 4y$$

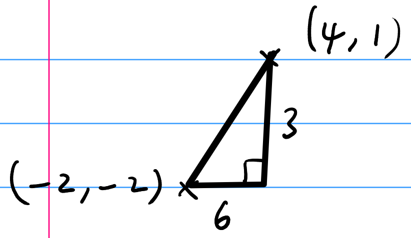
$$(4y)^2 - 2(4y)y + 4y^2 - 12 = 0$$

$$16y^2 - 8y^2 + 4y^2 = 12$$

$$12y^2 = 12$$
$$y^2 = 1$$
$$y = \pm 1$$

at P $y > 0$ $\therefore$ $y = 1$ $\quad x = 4$

$(4, 1)$



$$PQ = \sqrt{6^2 + 3^2}$$
$$= \underline{3\sqrt{5}}$$

70 Differentiate with respect to x:

(a) $\sin(2x + 5)$

(b) $\dfrac{3x^2}{2x - 1}$

(c) $\tan(x^2)$

a/ $2\cos(2x+5)$

b/ $u = 3x^2 \qquad v = 2x - 1$

$\frac{du}{dx} = 6x \qquad \frac{dv}{dx} = 2$

$$\frac{6x(2x-1) - 6x^2}{(2x-1)^2}$$

$$\frac{12x^2 - 6x - 6x^2}{(2x-1)^2}$$

$$\frac{6x^2 - 6x}{(2x-1)^2}$$

c/ $2x\sec^2(x^2)$

71 The curve C is defined by the parametric equations $x = 2\cos t$, $y = \sin t$.

The line L is a tangent to C at the point given by $t = \dfrac{\pi}{3}$

Find the point where L cuts the y-axis.

$\frac{dx}{dt} = -2\sin t \qquad \frac{dy}{dt} = \cos t$

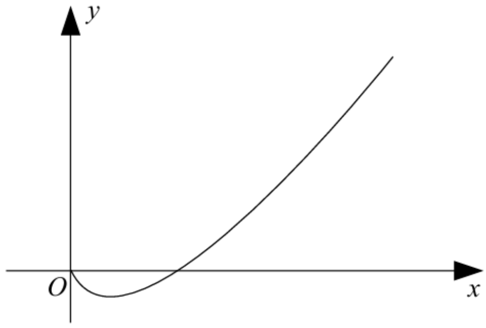
$\frac{dy}{dx} = \frac{\cos t}{-2\sin t}$ when $t = \frac{\pi}{3}$ $\frac{dy}{dx} = -\frac{\sqrt{3}}{6}$

when $t = \frac{\pi}{3}$ $x = 1$ $y = \frac{\sqrt{3}}{2}$

$$y - \frac{\sqrt{3}}{2} = \frac{-\sqrt{3}}{6}(x - 1)$$

$$y = \frac{-\sqrt{3}}{6}x + \frac{2\sqrt{3}}{3}$$

cuts y at $\frac{2\sqrt{3}}{3}$



The diagram shows the curve given parametrically by the equations $x = 4t^2$, $y = t(t^2 - 8)$, for $t > 0$

(a) Show that $\dfrac{dy}{dx} = \dfrac{3t^2 - 8}{8t}$ **(3)**

(b) Find the coordinates of the point on the curve at which the tangent to the curve is parallel to the line $3x - 12y + 8 = 0$ **(3)**

(c) Find the cartesian equation of the curve. **(3)**

$$\frac{dx}{dt} = 8t \qquad y = t^3 - 8t$$

$$\frac{dy}{dt} = 3t^2 - 8$$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$= \frac{3t^2 - 8}{8t}$$

b/ $3x - 12y + 8 = 0$

$$3x + 8 = 12y$$

$$y = \frac{3}{12}x + \frac{8}{12}$$

$$= \frac{1}{4}x + \frac{2}{3}$$

parallel when $\dfrac{dy}{dx} = \dfrac{1}{4}$

$$\frac{3t^2 - 8}{8t} = \frac{1}{4}$$

$$3t^2 - 8 = 2t$$

$$3t^2 - 2t - 8 = 0$$

$$t = 2 \qquad t = -\frac{4}{3}$$

$t > 0 \quad \therefore \quad t = 2$

when $t = 2$ $\quad x = 4(2)^2 \quad\quad y = 2((2)^2 - 8)$

$= 16 \quad\quad = -8$

$$\underline{\underline{(16, -8)}}$$

c/ $x = 4t^2 \quad\quad y = t^3 - 8t$

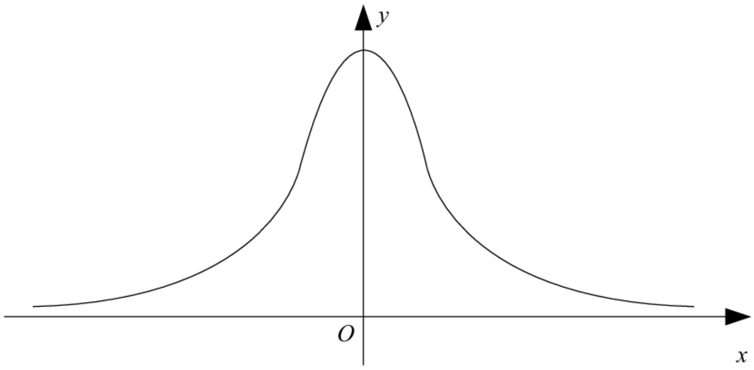
$t = \sqrt{\frac{x}{4}}$

$$y = \left(\sqrt{\frac{x}{4}}\right)^3 - 8\sqrt{\frac{x}{4}}$$

$$= \left(\frac{x^{\frac{1}{2}}}{2}\right)^3 - 8\,\frac{x^{\frac{1}{2}}}{2}$$

$$\underline{\underline{y = \frac{x^{\frac{3}{2}}}{8} - 4x^{\frac{1}{2}}}}$$

73



The diagram shows the curve with the equation $y = \frac{1}{1+x^2}$

(a) Find $\frac{d^2y}{dx^2}$

(b) Hence find the set of values of x for which the curve is concave.

a/ $y = (1 + x^2)^{-1}$

$$\frac{dy}{dx} = -2x(1 + x^2)^{-2}$$

$u = -2x \qquad v = (1 + x^2)^{-2}$

$\frac{du}{dx} = -2 \qquad \frac{dv}{dx} = -4x(1 + x^2)^{-3}$

$$\frac{d^2y}{dx^2} = -2(1 + x^2)^{-2} + 8x^2(1 + x^2)^{-3}$$

$$= \frac{-2}{(1 + x^2)^2} + \frac{8x^2}{(1 + x^2)^3}$$

b/ concave when $\frac{d^2y}{dx^2} < 0$

$$\frac{8x^2}{(1 + x^2)^3} - \frac{2}{(1 + x^2)^2} < 0$$

$$8x^2 - 2(1 + x^2) < 0$$

$$8x^2 - 2 - 2x^2 < 0$$

$$6x^2 - 2 < 0$$

$$3x^2 - 1 < 0$$

$$-\frac{\sqrt{3}}{3} < x < \frac{\sqrt{3}}{3}$$