1 Work out the first four terms, in ascending powers of x, in the binomial expansion of $\frac{1}{(1+x)^2}$

$$(1+x)^{-2}$$

$$1 + (-2)x + \frac{(-2)(-3)}{2}x^2 + \frac{(-2)(-3)(-4)}{6}x^3$$

$$1 - 2x + 3x^2 - 4x^3 \quad , |x| < 1$$

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2 (a) Expand $(4+6x)^{-\frac{1}{2}}$ in ascending powers of x up to and including x^3

(b) Write down the range of values of x for which the expansion is valid.

a/ $$4^{-\frac{1}{2}}\left(1+\frac{3}{2}x\right)^{-\frac{1}{2}}$$

$$\frac{1}{2}\left(1 + \left(-\frac{1}{2}\right)\left(\frac{3}{2}x\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(\frac{3}{2}x\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{6}\left(\frac{3}{2}x\right)^3\right)$$

$$\frac{1}{2}\left(1 - \frac{3}{4}x + \frac{27}{8}x^2 - \frac{135}{128}x^3\right)$$

$$\frac{1}{2} - \frac{3}{8}x + \frac{27}{16}x^2 - \frac{135}{256}x^3$$

b/ $$\left|\frac{3}{2}x\right| < 1$$
$$|x| < \frac{2}{3}$$

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3 (a) Expand $(8+12x)^{\frac{1}{3}}$ in ascending powers of x up to and including x^3

(b) Write down the range of values of x for which the expansion is valid.

a/ $$8^{\frac{1}{3}}\left(1+\frac{3}{2}x\right)^{\frac{1}{3}}$$

$$2\left(1 + \left(\frac{1}{3}\right)\left(\frac{3}{2}x\right) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}\left(\frac{3}{2}x\right)^2 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{6}\left(\frac{3}{2}x\right)^3\right)$$

$$2\left(1 + \frac{1}{2}x - \frac{1}{4}x^2 + \frac{5}{24}x^3\right)$$

$$2 + x - \frac{1}{2}x^2 + \frac{5}{12}x^3$$

b/ $$\left|\frac{3}{2}x\right| < 1$$
$$x < \left|\frac{2}{3}\right|$$

4 (a) Expand $\sqrt[3]{1+2x}$ in ascending powers of x up to and including x^3 and state the set of values of x for which the expansion is valid. **(5)**

(b) Use your expansion to find an approximation for $\sqrt[3]{1.1}$ to 5 decimal places. **(2)**

a) $(1+2x)^{\frac{1}{3}}$

$1 + \left(\frac{1}{3}\right)(2x) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}(2x)^2 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{6}(2x)^3$

$1 + \frac{2}{3}x - \frac{4}{9}x^2 + \frac{40}{81}x^3$ $\qquad |x| < \frac{1}{2}$

b) $1 + 2x = 1.1$

$x = 0.05$

$1 + \frac{2}{3}(0.05) - \frac{4}{9}(0.05)^2 + \frac{40}{81}(0.05)^3$

1.03228

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5 (a) Work out the first three terms in the binomial expansion of $\frac{1}{(2+3x)^2}$

(b) Write down the range of values of x for which the expansion is valid.

a) $(2+3x)^{-2}$

$2^{-2}\left(1+\frac{3}{2}x\right)^{-2}$

$\frac{1}{4}\left(1 + (-2)\left(\frac{3}{2}x\right) + \frac{(-2)(-3)}{2}\left(\frac{3}{2}x\right)^2\right)$

$\frac{1}{4}\left(1 - 3x + \frac{27}{4}x^2\right)$

$\frac{1}{4} - \frac{3}{4}x + \frac{27}{16}x^2$

b) $\left|\frac{3}{2}x\right| < 1$

$|x| < \frac{2}{3}$

6 (a) Expand $(3+2x)^{\frac{1}{2}}$ in ascending powers of x up to and including x^3

(b) Write down the range of values of x for which the expansion is valid.

(c) Use your expansion, with $x = 0.1$, to find an approximation of the value of $\sqrt{5}$
Give your answer to 3 decimal places.

a/ $3^{\frac{1}{2}}\left(1+\frac{2}{3}x\right)^{\frac{1}{2}}$

$\sqrt{3}\left(1+\frac{1}{2}\left(\frac{2}{3}x\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}\left(\frac{2}{3}x\right)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}\left(\frac{2}{3}x\right)^3\right)$

$\sqrt{3}\left(1+\frac{1}{3}x-\frac{1}{18}x^2+\frac{1}{54}x^3\right)$

$\sqrt{3}+\frac{\sqrt{3}}{3}x-\frac{\sqrt{3}}{18}x^2+\frac{\sqrt{3}}{54}x^3$

b/ $\left|\frac{2}{3}x\right|<1$

$|x|<\frac{3}{2}$

c/ $\sqrt{3.2}=\frac{4\sqrt{5}}{5}$

$\frac{4\sqrt{5}}{5}=\sqrt{3}+\frac{\sqrt{3}}{3}(0.1)-\frac{\sqrt{3}}{18}(0.1)^2+\frac{\sqrt{3}}{54}(0.1)^3$

$\frac{4}{5}\sqrt{5}=1.788855\ldots$

$\sqrt{5}=\underline{\underline{2.236}}$

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7 (a) Expand $(1+3x)^{-3}$ in ascending powers of x up to and including x^3

(b) Write down the range of values of x for which the expansion is valid.

(c) Use your expansion to find the expansion of $\frac{2x+1}{(1+3x)^3}$ up to and including x^3

a/ $1+(-3)(3x)+\frac{(-3)(-4)}{2}(3x)^2+\frac{(-3)(-4)(-5)}{6}(3x)^3$

$\underline{\underline{1-9x+54x^2-270x^3}}$

b/ $|3x| < 1$

$|x| < \frac{1}{3}$

c/ $(2x+1)(1 - 9x + 54x^2 - 270x^3)$

$2x + 1 - 18x^2 - 9x + 108x^3 + 54x^2 - 540x^4 - 270x^3$

$\underline{\underline{1 - 7x + 36x^2 - 162x^3}}$

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8 (a) Expand $(9-2x)^{\frac{1}{2}}$ in ascending powers of x up to and including x^3

(b) Write down the range of values of x for which the expansion is valid.

(c) Use your expansion, with a suitable value of x, to find the value of $\sqrt{8.9}$ correct to 5 significant figures

a/ $9^{\frac{1}{2}}\left(1 - \frac{2}{9}x\right)^{\frac{1}{2}}$

$3\left(1 + \left(\frac{1}{2}\right)\left(-\frac{2}{9}x\right) + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)}{2}\left(-\frac{2}{9}x\right)^2 + \frac{\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)}{6}\left(\frac{-2}{9}x\right)^3\right)$

$3\left(1 - \frac{1}{9}x - \frac{1}{162}x^2 - \frac{1}{1458}x^3\right)$

$3 - \frac{1}{3}x - \frac{1}{54}x^2 - \frac{1}{486}x^3$

b/ $|2x| < 1$

$|x| < \frac{1}{2}$

c/ $9 - 2x = 8.9$

$2x = 0.1$

$x = 0.05$

$3 - \frac{1}{3}(0.05) - \frac{1}{54}(0.05)^2 - \frac{1}{486}(0.05)^3$

$\underline{\underline{2.9833}}$

9 (a) Use binomial expansion to show that $\sqrt{\frac{1+5x}{1-x}} \approx 1 + 3x - \frac{3}{2}x^2$

A student substitutes $x = \frac{1}{2}$ into the expansion to find an estimate for $\sqrt{7}$

(b) Give a reason why the student should not use $x = \frac{1}{2}$

(c) Substitute $x = \frac{1}{9}$ into $\sqrt{\frac{1+5x}{1-x}} = 1 + 3x - \frac{3}{2}x^2$

to obtain an approximation for $\sqrt{7}$. Give your answer as a fraction in its simplest form.

$$\frac{(1+5x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}} = (1+5x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$$

$$(1+5x)^{\frac{1}{2}} = 1 + \tfrac{1}{2}(5x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(5x)^2$$

$$= 1 + \frac{5}{2}x - \frac{25}{8}x^2$$

$$(1-x)^{-\frac{1}{2}} = 1 + (-\tfrac{1}{2})(-x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(-x)^2$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2$$

$$\left(1 + \frac{5}{2}x - \frac{25}{8}x^2\right)\left(1 + \frac{1}{2}x + \frac{3}{8}x^2\right)$$

$$1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{2}x + \frac{5}{4}x^2 + \ldots - \frac{25}{8}x^2 + \ldots$$

$$\underline{1 + 3x - \frac{3}{2}x^2}$$

b/ $(1+5x)^{\frac{1}{2}}$ expansion is valid where $|5x| < 1$

$\underline{|x| < \frac{1}{5}}$

c/ $x = \frac{1}{9}$ $\quad \sqrt{\frac{1+5(\frac{1}{9})}{1-\frac{1}{9}}} = \frac{\sqrt{7}}{2}$

$$\frac{\sqrt{7}}{2} = 1 + 3\left(\tfrac{1}{9}\right) - \tfrac{3}{2}\left(\tfrac{1}{9}\right)^2$$

$$= \frac{71}{54}$$

$$\sqrt{7} = \frac{71}{27}$$

10 (a) Find the first three terms, in ascending powers of x, of the binomial expansion of

$$\frac{1}{\sqrt{9+x}}$$

giving each coefficient in its simplest form.

The expansion can be used to find an approximation of $\sqrt{3}$

Possible values of x that could be substituted into this expansion are

- $x = 18$ because $\dfrac{1}{\sqrt{9+18}} = \dfrac{1}{\sqrt{27}} = \dfrac{\sqrt{3}}{9}$
- $x = 3$ because $\dfrac{1}{\sqrt{9+3}} = \dfrac{1}{\sqrt{12}} = \dfrac{\sqrt{3}}{6}$
- $x = -\dfrac{2}{3}$ because $\dfrac{1}{\sqrt{9-\dfrac{2}{3}}} = \dfrac{1}{\sqrt{\dfrac{25}{3}}} = \dfrac{\sqrt{3}}{5}$

(b) Without evaluating your expansion,

(i) state, giving a reason, which of the three values of x should not be used

(ii) state, giving a reason, which of the three values of x would lead to the most accurate approximation to $\sqrt{3}$

$$(9+x)^{-\frac{1}{2}}$$

$$9^{-\frac{1}{2}}\left(1+\frac{1}{9}x\right)^{-\frac{1}{2}}$$

$$\frac{1}{3}\left(1+\left(-\frac{1}{2}\right)\left(\frac{1}{9}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(\frac{1}{9}x\right)^2\right)$$

$$\frac{1}{3}\left(1-\frac{1}{18}x+\frac{1}{216}x^2\right)$$

$$\frac{1}{3}-\frac{1}{54}x+\frac{1}{648}x^2$$

b/ $\left|\frac{1}{9}x\right| < 1$

$|x| < 9$

i/ x cannot be 18 as the expansion is valid for $|x| < 9$

ii/ $-\frac{2}{3}$ because it is closest to zero.

11 (a) Find the first four terms, in ascending powers of x, of the binomial expansion of $\sqrt{1+6x}$ giving each term in its simplest form. **(3)**

(b) Explain how you could use $x = \frac{1}{12}$ in the expansion to find an approximation for $\sqrt{6}$

There is no need to carry out the calculation. **(2)**

a/ $(1+6x)^{\frac{1}{2}} = 1 + (\frac{1}{2})(6x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(6x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{6}(6x)^3$

$= \underline{1 + 3x - \frac{9}{2}x^2 + \frac{27}{2}x^3}$

b/ $\sqrt{1+6(\frac{1}{12})} = \frac{\sqrt{6}}{2}$

Substitute $x = \frac{1}{12}$ into the expansion, then multiply the answer by 2.

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12 (a) Find the first three terms, in ascending powers of x, of the binomial expansion of

$$\frac{1}{\sqrt{4+x}}$$

giving each coefficient in its simplest form.

(b) Hence find first three terms, in ascending powers of x, of the binomial expansion of

$$\frac{1}{\sqrt{4-x^2}}$$

a/ $(4+x)^{-\frac{1}{2}}$

$4^{-\frac{1}{2}}(1+\frac{1}{4}x)^{-\frac{1}{2}}$

$\frac{1}{2}\left(1 + (-\frac{1}{2})(\frac{1}{4}x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(\frac{1}{4}x)^2\right)$

$\frac{1}{2}(1 - \frac{1}{8}x + \frac{3}{128}x^2)$

$\frac{1}{2} - \frac{1}{16}x + \frac{3}{256}x^2$

b/ $\frac{1}{2} + \frac{1}{16}x^2 + \frac{3}{256}x^4$

13 (a) Find the first three terms of the expansion of $(9+2x)^{\frac{-1}{2}}$ in ascending powers of x.

(b) The expansion of $\dfrac{a+bx}{\sqrt{9+2x}}$ is $3+x\ldots$ Find the values of the constants a and b.

a/ $(9+2x)^{-\frac{1}{2}}$

$9^{-\frac{1}{2}}\left(1+\frac{2}{9}x\right)^{-\frac{1}{2}}$

$\frac{1}{3}\left(1+\frac{2}{9}x\right)^{-\frac{1}{2}} = \frac{1}{3}\left(1+\left(-\frac{1}{2}\right)\left(\frac{2}{9}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(\frac{2}{9}x\right)^2\right)$

$= \frac{1}{3}\left(1-\frac{1}{9}x+\frac{1}{54}x^2\right)$

$= \frac{1}{3}-\frac{1}{27}x+\frac{1}{162}x^2$

b/ $(a+bx)\left(\frac{1}{3}-\frac{1}{27}x+\frac{1}{162}x^2\right)$

$\frac{1}{3}a+\frac{1}{3}bx-\frac{1}{27}ax\ldots$

$\frac{1}{3}a=3$

$\underline{\underline{a=9}}$

$\frac{1}{3}bx-\frac{1}{27}ax = x$

$\frac{1}{3}b-\frac{1}{27}a=1$

$\frac{1}{3}b-\frac{1}{3}=1$

$\underline{\underline{b=4}}$

14 (a) Find the coefficient of x^5 in the expansion of $(2+3x)^7$

(b) (i) Expand $\sqrt{1-2x}$ as far as the term x^3.

(ii) State the range of the values for which your expansion is valid.

(iii) Use your expansion to find an estimate for $\sqrt{98}$, correct to five decimal places, and compare this with the value of $\sqrt{98}$ given by your calculator.

$$(2+3x)^7 = 2^7\left(1+\frac{3}{2}x\right)^7$$

$$1+nx+\frac{n(n-1)}{2}x^2+\frac{n(n-1)(n-2)}{6}x^3+\ldots+\frac{n(n-1)(n-2)(n-3)(n-4)}{5!}x^5$$

$$128\times\frac{7\times6\times5\times4\times3}{120}\left(\frac{3}{2}x\right)^5 = 20412x^5$$

$\underline{\underline{20412}}$

b/
$$(1-2x)^{\frac{1}{2}} = 1+\left(\tfrac{1}{2}\right)(-2x)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}(-2x)^2+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}(-2x)^3$$

$$= \underline{\underline{1-x-\frac{1}{2}x^2-\frac{1}{2}x^3}}$$

ii/ $|2x|<1$

$\underline{\underline{|x|<\frac{1}{2}}}$

iii/ $\sqrt{98} = \sqrt{2}\sqrt{49}$

$= 7\sqrt{2}$

let $x=0.01$ $\quad (1-2(0.01))^{\frac{1}{2}} = \frac{7\sqrt{2}}{10}$

$$1-(0.01)-\frac{1}{2}(0.01)^2-\frac{1}{2}(0.01)^3$$

$= 0.9899495$

$\therefore 7\sqrt{2} = \underline{\underline{9.89950}}$ $\qquad \sqrt{98} = 9.89949$ to 5dp

The estimate is very close to $\sqrt{98}$.

15 In this question you should assume $-\frac{1}{2} < x < \frac{1}{2}$

(a) For the binomial expansion of $(1-2x)^{-2}$ find and simplify the first four terms.

(b) Write down the sum to infinity of the series $1 + 2x + 4x^2 + 8x^3 + \dots$

(c) Hence or otherwise find and simplify an expression for $2 + 6x + 16x^2 + 40x^3 + \dots$

$\times 2 \quad \times 3 \quad \times 4 \quad \times 5$

a/ $(1-2x)^{-2}$

$$1 + (-2)(-2x) + \frac{(-2)(-3)}{2}(-2x)^2 + \frac{(-2)(-3)(-4)}{6}(-2x)^3$$

$$\underline{\underline{1 + 4x + 12x^2 + 32x^3}}$$

b/ $r = 2x \qquad a = 1$

$$S_\infty = \frac{a}{1-r} = \frac{1}{1-2x}$$

c/ part a + part b

$$(1-2x)^{-2} + \frac{1}{1-2x}$$

$$\frac{1}{(1-2x)^2} + \frac{1}{1-2x}$$

$$\frac{1}{(1-2x)^2} + \frac{1-2x}{(1-2x)^2}$$

$$\frac{2-2x}{(1-2x)^2}$$

16 Find the first three terms of the expansion of $(8-4x^3)^{\frac{1}{3}}$ in ascending powers of x.

$$(8-4x^3)^{\frac{1}{3}}$$

$$8^{\frac{1}{3}}\left(1-\frac{1}{2}x^3\right)^{\frac{1}{3}}$$

$$2\left(1+\left(\frac{1}{3}\right)\left(-\frac{1}{2}x^3\right)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}\left(-\frac{1}{2}x^3\right)^2\right)$$

$$2\left(1-\frac{1}{6}x^3-\frac{1}{36}x^6\right)$$

$$\underline{2-\frac{1}{3}x^3-\frac{1}{18}x^6}$$

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17 The function f(x) is given by $f(x)=\sqrt{1+ax}$ where a is a non-zero constant.

In the binomial expansion of f(x), the coefficients of x and x^2 are equal.

(a) Find the value of a.

(b) Using this value of a
  (i) State the set of values for which the binomial expansion is valid,
  (ii) Write down the quadratic function that approximates f(x) when x is small.

a) $$(1+ax)^{\frac{1}{2}} = 1+\left(\frac{1}{2}\right)ax+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}(ax)^2$$

$$= 1+\frac{1}{2}ax-\frac{1}{8}a^2x^2$$

$$\frac{1}{2}a=-\frac{1}{8}a^2$$

$$\frac{1}{8}a^2+\frac{1}{2}a=0$$

$$a^2+4a=0$$

$$a(a+4)=0$$

$$a=0 \quad \underline{\underline{a=-4}}$$

b i) $(1-4x)^{\frac{1}{2}}$ $\quad |4x|<1$

$$\underline{\underline{|x|<\frac{1}{4}}}$$

ii) $\underline{\underline{1-2x-2x^2}}$

18 (a) Work out the first three terms in the binomial expansion of $(1+3x)^{\frac{1}{2}}$

(b) Write down the range of values of x for which the expansion is valid.

a) $$(1+3x)^{\frac{1}{2}} = 1 + \left(\tfrac{1}{2}\right)(3x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}(3x)^2$$

$$= 1 + \frac{3}{2}x - \frac{9}{8}x^2$$

b) $$|3x| < 1$$

$$|x| < \frac{1}{3}$$