1 An arithmetic series has first term a and common difference d.
Prove that the sum of the first n terms of the series is

$$\frac{1}{2}n(2a+(n-1)d)$$

$$S_n = a + a+d + a+2d + \dots + a+(n-3)d + a+(n-2)d + a+(n-1)d$$
$$S_n = a+(n-1)d + a+(n-2)d + a+(n-3)d + \dots + a+2d + a+d + a$$

$$2S_n = 2a+(n-1)d + 2a+(n-1)d + 2a+(n-1)d + \dots + 2a+(n-1)d + 2a+(n-1)d + 2a+(n-1)d$$

$$2S_n = n(2a+(n-1)d)$$
$$S_n = \frac{1}{2}n(2a+(n-1)d)$$

---

2 The fifth term of an arithmetic sequence is 5 and the eighth term of the sequence is -16.

(a) Find the first term of the sequence.
(b) Find the common difference.

$U_5 = 5$ $\quad U_8 = -16$ $\quad U_n = a+(n-1)d$

$5 = a+4d$
$-16 = a+7d$ (Simultaneous equations)

$\underline{a = 33}$ $\quad \underline{d = -7}$

---

3 The third term of an arithmetic series is -4 and the sum of the first eight terms of the series is 22.

(a) Find the first term of the series.
(b) Find the common difference.
(c) Find the highest value of n for which the sum of the first n terms is less than 200.

$U_3 = -4$ $\quad S_8 = 22$ $\quad S_n = \frac{n}{2}(2a+(n-1)d)$

$-4 = a+2d$ $\quad 22 = \frac{8}{2}(2a+7d)$
$22 = 8a+28d$ Sim. eq.

$\underline{a = -13}$ $\quad \underline{d = 4.5}$

$$\frac{n}{2}(2a+(n-1)d) < 200$$

$$\frac{n}{2}(-26+(n-1)4.5) < 200$$

$$n(-26+4.5n-4.5) < 400$$

$$n(-30.5+4.5n) < 400$$

$$4.5n^2 - 30.5n < 400$$

$$4.5n^2 - 30.5n - 400 < 0$$

$$-6.6 < n < 13.4$$

$$\therefore \underline{\underline{n = 13}}$$

---

4 Bob saves some money every week. He saves £2.20 in the first week, £2.40 in the second week, £2.60 in the third week, and so on until week 100. His weekly savings form an arithmetic sequence.

(a) Find the amount he saves in week 100.
(b) Calculate his total savings over the 100 week period.

$a = 2.2 \qquad d = 0.2$

a) $U_{100} = a + 99d$

$= 2.2 + 99(0.2)$

$= 22$

$\underline{\underline{£22}}$

b) $S_{100} = \frac{100}{2}(2(2.2) + 99(0.2))$

$\underline{= £1210}$

5 The first three terms of an arithmetic series are (k + 3), (2k + 4) and (4k – 2) respectively.

(a) Find the value of the constant k.
(b) Find the sum of the first 20 terms of the series.

a/ $$u_3 - u_2 = u_2 - u_1$$

$$(4k-2) - (2k+4) = (2k+4) - (k+3)$$

$$2k - 6 = k + 1$$

$$k = 7$$

b/ $\therefore\ a = 7 + 3 = \underline{10}$ $\qquad u_2 = 2(7) + 4 = \underline{18}$

$a = 10 \qquad d = 8$

$$S_{20} = \frac{20}{2}\left(2(10) + 19(8)\right)$$

$$= \underline{1720}$$

---

6 The amount of cars produced by a factory each week forms an arithmetic sequence.
In the first week the factory produces 100 cars. The number of cars produced will increase by 4 each week until the number of cars being produced reaches 180.
The factory will then continue to produce 180 cars each week.

(a) After how many weeks does the factory reach production of 180 cars per week. **(2)**
(b) Find the total number of cars produced in the first 52 weeks. **(4)**

$a = 100 \qquad d = 4$

$$180 = 100 + (n-1)4$$

$$180 = 100 + 4n - 4$$

$$4n = 84$$

$\underline{n = 21}$ (the 21^{st} week)

$$S_{21} = \frac{21}{2}\left(2(100) + 20(4)\right) = 2940$$

Last 31 weeks: $31 \times 180 = 5580$

52 weeks: $2940 + 5580 = \underline{\underline{8520}}$

7 Bertie makes payments into a savings account every month. He pays in £300 in the first month and the amount he pays increases by £40 each subsequent month.
Charlotte also makes payments into a savings account. She pays in £500 in the first month and the amount she pays in increases by £20 each subsequent month

After how many months have Bertie and Charlotte paid in the same amount in total.

B: $a = 300 \quad d = 40$ C: $a = 500 \quad d = 20$

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

$$\frac{n}{2}(2(300) + (n-1)40) = \frac{n}{2}(2(500) + (n-1)20)$$

$$600 + 40n - 40 = 1000 + 20n - 20$$
$$40n + 560 = 980 + 20n$$
$$20n = 420$$
$$n = 21$$

8 A geometric series is $a + ar + ar^2 + \ldots$

Prove that the sum of the first n terms of the series is

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_n = a + ar + ar^2 + \ldots + ar^{n-3} + ar^{n-2} + ar^{n-1} \quad ①$$
$$rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} + ar^n \quad ②$$

① - ② $$S_n - rS_n = a - ar^n$$
$$S_n(1-r) = a(1-r^n)$$
$$S_n = \frac{a(1-r^n)}{1-r}$$

9 The fifth term of a geometric series is 12 and the eighth term of the series is 96.

(a) Find the common ratio.
(b) Find the first term of the series.
(c) The sum of the first 20 terms, giving your answer to the nearest whole number.

$U_5 = 12 \qquad U_8 = 96 \qquad U_n = ar^{n-1}$

a/ $12 = ar^4 \; (1) \qquad 96 = ar^7 \; (2)$

$(2) \div (1) \qquad \frac{96}{12} = \frac{ar^7}{ar^4}$

$8 = r^3$

$\underline{r = 2}$

b/ $12 = a(2)^4$

$12 = 16a$

$a = \frac{3}{4}$

c/ $S_n = \frac{a(1-r^n)}{1-r}$

$S_{20} = \frac{\frac{3}{4}(1-2^{20})}{1-2}$

$= \underline{\underline{786431.25}}$

---

10 The third term of a geometric series is 135 and the sixth term of the series is 40.

(a) Find the common ratio.
(b) Find the first term of the series.
(c) The sum of the first 10 terms of the series.
(d) The sum to infinity of the series.

a/ $ar^2 = 135 \; (1) \qquad ar^5 = 40 \; (2) \qquad (2) \div (1)$

$\frac{ar^5}{ar^2} = \frac{40}{135}$

$$r^3 = \frac{8}{27}$$

$$r = \frac{2}{3}$$

b) $$a\left(\frac{2}{3}\right)^2 = 135$$

$$a = 303.75$$

c) $$S_{10} = \frac{303.75\left(1-\left(\frac{2}{3}\right)^{10}\right)}{1-\frac{2}{3}}$$

$$= 895.4 \quad (1dp)$$

d) $$S_\infty = \frac{a}{1-r} = \frac{303.75}{1-\frac{2}{3}}$$

$$= 911.25$$

11 The second term of a geometric series is 3.75 and the sum to infinity is 20.

(a) Find the two possible values of r.
(b) Find the corresponding two possible values of a.
Given that r takes the larger of the two possible values,
(c) Find the smallest value of n for which $S_n > 19$

$$3.75 = ar \qquad 20 = \frac{a}{1-r}$$

$$20 - 20r = a$$

$$3.75 = r(20-20r)$$

$$3.75 = 20r - 20r^2$$

$$20r^2 - 20r + 3.75 = 0$$

$$r = \frac{3}{4} \text{ or } r = \frac{1}{4}$$

b/ $a = \frac{3.75}{3/4}$ $a = \frac{3.75}{1/4}$

$\underline{a = 5}$ or $\underline{a = 15}$

c/ $r = \frac{3}{4}$

$$19 = \frac{5\left(1 - \frac{3}{4}^n\right)}{1 - \frac{3}{4}}$$

$$\frac{19}{20} = 1 - \left(\frac{3}{4}\right)^n$$

$$\left(\frac{3}{4}\right)^n = \frac{1}{20}$$

$$n = \log_{0.75} \frac{1}{20}$$
$$= 10.4$$

$\therefore \underline{\underline{n = 11}}$

---

12 The first three terms of a geometric series are $(2k - 2)$, $(k + 3)$, and k respectively, where k is a positive constant.

(a) Show that $k^2 - 8k - 9 = 0$. **(4)**
(b) Hence show that $k = 9$. **(2)**
(c) Find the common ratio. **(2)**
(d) The sum to infinity of the series. **(2)**

a/ $$\frac{k+3}{2k-2} = \frac{k}{k+3}$$

$$(k+3)(k+3) = k(2k-2)$$

$$k^2 + 6k + 9 = 2k^2 - 2k$$
$$0 = k^2 - 8k - 9$$

b/ $$0 = (k-9)(k+1)$$

$\underline{\underline{k = 9}}$ $\cancel{k = -1}$ k is a positive constant

$\therefore k = 9$

c/ $a = 2(9) - 2$ $\quad u_2 = 9 + 3$ $\quad u_3 = k$

$= 16$ $\quad = 12$ $\quad = 9$

$r = \frac{12}{16} = \underline{\underline{\frac{3}{4}}}$

d/ $S_\infty = \frac{16}{1 - \frac{3}{4}}$

$= \underline{\underline{64}}$

---

13 Sophie will be paid a salary of £35000 in 2018. Each year Sophie will get a 3% pay rise, the first increase being in 2019, so that her salaries form a geometric sequence

(a) Find, to the nearest £100, Sophie's salary in 2020.
Sophie will receive a salary each year until she retires at the end of 2037. **(2)**
(b) Find, to the nearest £100, the total amount Sophie will have earned from 2018 until she reties in 2037. **(4)**

a/ $a = 35000$ $\quad r = 1.03$

$u_3 = 35000 \cdot 1.03^2$

$= £37100$ (nearest £100)

b/ 20 years $\quad S_{20} = \frac{35000(1 - 1.03^{20})}{1 - 1.03}$

$= \underline{\underline{£940500}}$

14 A sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_1 = 2$$
$$x_{n+1} = ax_n - 5, \quad n \geq 1$$

(a) Find an expression for x_2 in terms of a
(b) Show that $x_3 = 2a^2 - 5a - 5$
Given that $x_3 = 20$
(c) Find the possible values of a.

a/ $x_2 = a(x_1) - 5$
$= \underline{2a - 5}$

b/ $x_3 = a(2a - 5) - 5$
$= 2a^2 - 5a - 5$

c/ $2a^2 - 5a - 5 = 20$
$2a^2 - 5a - 25 = 0$
$(2a+5)(a-5)$
$\underline{a = -\frac{5}{2}}$ $\underline{\underline{a = 5}}$

---

15 A sequence $x_1, x_2, x_3, \ldots$ is defined by

$$x_1 = k$$
$$x_n = 4x_{n-1} - 1, \quad n \geq 2$$

(a) Find an expression for x_2 in terms of k
(b) Show that $x_3 = 16k - 5$
(c) Find $\sum_{r=1}^{4} a_r$ in terms of k

a/ $x_2 = 4(x_1) - 1$
$= 4k - 1$

b/ $x_3 = 4(4k-1) - 1$
$= 16k - 5$

c/ $x_4 = 4(16k - 5) - 1$
$= 64k - 21$
$\sum_{r=1}^{4} a_r = k + 4k - 1 + 16k - 5 + 64k - 21$
$= \underline{85k - 27}$

16 A sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 1$$
$$u_{n+1} = (u_n - 1)^2, \qquad n \geq 1$$

(a) Find u_2, u_3 and u_4
(b) Write down the value of u_{100}

a/ $u_2 = (u_1 - 1)^2$
$= (1-1)^2$
$= 0$
$u_3 = (0-1)^2$
$= 1$
$u_4 = 0$

b/ $u_{100} = 0$ (all even terms are zero)

---

17 A sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 3$$
$$u_{n+1} = u_n + c, \quad n \geq 1$$

Given $u_5 = 21$
(a) Find the value of c
(b) Find an expression for u_n in terms of n

a/ $u_2 = u_1 + c$
$= 3 + c$
$u_3 = 3 + 2c$
$u_4 = 3 + 3c$
$u_5 = 3 + 4c$

$21 = 3 + 4c$
$18 = 4c$
$c = 4.5$

b/ $3, 7.5, 12, 16.5, 21 \ldots$

$u_n = 4.5n - 1.5$

18 A sequence $u_1, u_2, u_3, \ldots$ is defined by

$$u_1 = 3, u_2 = 5$$
$$u_n = u_{n-1} + u_{n-2}, \qquad n \geq 3$$

Find u_3, u_4 and u_5

$u_3 = u_2 + u_1$
$= \underline{8}$
$u_4 = u_3 + u_2$
$= 8 + 5$
$= 13$
$u_5 = 13 + 8$
$= \underline{21}$

---

19 The value, £V, of a car t years after it was bought on January 1 2015 is modelled by the equation $V = Ak^t$ where A and k are constants.

Given that the car was worth £20900 on January 1 2019 and £14200 on January 1 2022

(a) (i) Find k to 3 decimal places.
(ii) Show that A is approximately £35000 **(4)**

(b) With reference to the model, interpret
(i) The value of the constant A
(ii) The value of k **(2)**

(c) Using the model, find the year during which the value of the car first falls below £5000 **(4)**

a) $V = Ak^t$

$20900 = Ak^4$ (1) $\qquad 14200 = Ak^7$ (2)

(2) ÷ (1) $\qquad \dfrac{14200}{20900} = \dfrac{Ak^7}{Ak^4}$

$\dfrac{142}{209} = k^3$

$k = \sqrt[3]{\dfrac{142}{209}}$
$= \underline{0.879}$

$20900 = A(0.879)^4$
$A = \underline{\underline{34991}}$ $\quad \therefore$ approx. £35000

b/ i/ The car's initial value was £35000

ii/ Each year the value declines by 12.1%

c/ $5000 = 35000(0.879)^x$

$\frac{1}{7} = 0.879^x$

$x = \log_{0.879}\left(\frac{1}{7}\right)$

$= 15.1$

The 16^{th} year

---

20 (a) Show that $\sum_{r=1}^{12}(3r+7+2^r) = 8508$

(b) A sequence $u_1, u_2, u_3, \ldots$ is defined by $u_{n+1} = \frac{1}{u_n}$, $u_1 = \frac{2}{5}$

Find the exact value of $\sum_{r=1}^{100} u_r$

$3r+7$

$a = 10 \quad d = 3$

$S_{12} = \frac{12}{2}(2(10) + 11(3))$

$= 318$

2^r

$a = 2 \quad r = 2$

$S_{12} = \frac{2(1-2^{12})}{1-2}$

$= 8190$

$318 + 8190 = 8508$

21 An employee's salary is £25000 in the year 2023.
The salary is modelled to increase by 4% in each year after 2023.

Using the model,

(a) show that the salary is £27040 in 2025 **(2)**

(b) find the total salary the employee will earn until the end of 2040 **(4)**

a/ $25000 \times 1.04^2 = £27040$

b/ $a = 25000 \qquad r = 1.04$

$$S_{18} = \frac{25000(1 - 1.04^{18})}{1 - 1.04}$$

$$= \underline{\underline{£641135}} \text{ (nearest £)}$$

---

22 (a) Find the value of $\sum_{r=5}^{\infty} 27 \times \left(\frac{1}{3}\right)^r$

(b) Show that $\sum_{n=1}^{63} \log_2\left(\frac{n+1}{n}\right) = 6$

a/ $a = 27 \times \left(\frac{1}{3}\right)^5 = \frac{1}{9}$

$r = \frac{1}{3}$

$$S_\infty = \frac{a}{1 - r}$$

$$= \frac{1/9}{1 - 1/3}$$

$$= \underline{\underline{\frac{1}{6}}}$$

b/ $\log_2 \frac{2}{1} + \log_2 \frac{3}{2} + \log_2 \frac{4}{3} + \log_2 \frac{5}{4} + \ldots + \log_2 \frac{64}{63}$

$\log_2 \frac{2 \times 3 \times 4 \times 5 \times \ldots \times 63 \times 64}{1 \times 2 \times 3 \times 4 \times \ldots \times 62 \times 63} = \log_2 64 = \underline{\underline{6}}$

$u_1 = 40 \quad u_4 = 55$

23 Harry saved money each month in 2022. In January 2022 he saved £40. In April 2022 he saved £55.

Given the amount he saved each month is modelled by an **arithmetic series**,
(a) Find the total amount Harry saved in 2022. **(3)**

Given instead the amount he saved each month is modelled by a **geometric series**,
(b) Find the total amount Harry saved in 2022. **(3)**

a/ $a = 40 \quad d = 5$

$S_{12} = \frac{12}{2}\left(2(40) + 11(5)\right)$

$= £810$

b/ $a = 40 \qquad r = \sqrt[3]{\frac{55}{40}} = 1.11$

$u_4 = ar^3$

$55 = 40r^3$

$r^3 = \frac{55}{40}$

$S_{12} = \frac{40(1 - 1.11^{12})}{1 - 1.11}$

$= £920$ (nearest £)

---

24 A sequence $a_1, a_2, a_3, \ldots$ is defined by $a_{n+1} = \frac{k - a_n}{a_n}$

Where k is a constant.

Given that the sequence is periodic of order 3 and that $a_1 = 3$

(a) Show that $k^2 - 11k - 12 = 0$

(b) For this sequence explain why $k \neq 12$

(c) Find the value of $\sum_{r=1}^{100} a_r$

$a_4 = a_1 \qquad a_2 = \frac{k-3}{3}$

$a_3 = \frac{k - \frac{(k-3)}{3}}{\frac{k-3}{3}} = \frac{3}{k-3}\left(k - \frac{k-3}{3}\right)$

$= \frac{3k}{k-3} - 1$

$a_4 = \frac{k - \left(\frac{3k}{k-3} - 1\right)}{\frac{3k}{k-3} - 1}$

$$\frac{k - \frac{3k}{k-3} + 1}{\frac{3k}{k-3} - 1} = 3$$

$$k - \frac{3k}{k-3} + 1 = 3\left(\frac{3k}{k-3} - 1\right)$$

$$k - \frac{3k}{k-3} + 1 = \frac{9k}{k-3} - 3$$

$$k(k-3) - 3k + k - 3 = 9k - 3(k-3)$$
$$k^2 - 3k - 2k - 3 = 9k - 3k + 9$$
$$k^2 - 5k - 3 = 6k + 9$$
$$k^2 - 11k - 12 = 0$$

b/ $a_2 = \frac{k - a_1}{a_1}$

if $k = 12$ $\quad a_2 = \frac{12-3}{3} = 3$

every term would be 3 and therefore not periodic order 3.

c/ $k^2 - 11k - 12 = 0$

$(k-12)(k+1) = 0$

$k = 12 \quad \underline{\underline{k = -1}}$

$a_1 = 3 \quad a_2 = \frac{-1-3}{3} = \frac{-4}{3} \quad a_3 = \frac{-1 - \frac{-4}{3}}{-\frac{4}{3}}$

$= -\frac{1}{4}$

$$\sum_{r=1}^{r=100} a_r = 34(3) + 33\left(\frac{-4}{3}\right) + 33\left(\frac{-1}{4}\right)$$

$= \underline{\underline{49.75}}$

25 A periodic sequence is defined by $U_n = \cos(n\pi)$

State the period of this sequence.

$\underline{\underline{2}}$

---

26 An arithmetic sequence has first term a and common difference d.
The sum of the first 25 terms of the sequence is equal to the square of the sum of the first 10 terms.

(a) Show that $4a^2 + 36ad + 81d^2 = a + 12d$ **(4)**

(b) Given that the 10th term of the sequence is 2, find the possible values of a. **(5)**

a/ $S_{25} = \frac{25}{2}(2a + 24d)$

$S_{10} = \frac{10}{2}(2a + 9d)$

$$\frac{25}{2}(2a + 24d) = \left(5(2a+9d)\right)^2$$

$$25a + 300d = (10a + 45d)^2$$

$$25a + 300d = 100a^2 + 900ad + 2025d^2$$

$$a + 12d = 4a^2 + 36ad + 81d^2$$

b/ $U_{10} = 2$ $\quad a + 9d = 2$

$a = 2 - 9d$

$$2 - 9d + 12d = 4(2-9d)^2 + 36d(2-9d) + 81d^2$$

$$2 + 3d = 4(4 - 36d + 81d^2) + 72d - 324d^2 + 81d^2$$

$$2 + 3d = 16 - 144d + 324d^2 + 72d - 324d^2 + 81d^2$$

$$2 + 3d = 16 - 72d + 81d^2$$

$$0 = 81d^2 - 75d + 14$$

$\underline{d = \frac{2}{3}}$ or $\underline{d = \frac{7}{27}}$

$\underline{\underline{a = -4}}$ or $\underline{\underline{a = -\frac{1}{3}}}$

27 An arithmetic sequence has first term a and common difference d.
The sum of the first 8 terms of the sequence is 198

(a) Show that $4a + 14d = 99$

(b) Given that the sum of the first 25 terms is 300, find the sum of the first 21 terms.

(c) S_n is the sum of the first n terms of the sequence.
Explain why the value you found in part (b) is the maximum value of S_n

a/ $S_8 = 198$

$$\frac{8}{2}(2a + 7d) = 198$$

$$8a + 28d = 198$$

$$\underline{4a + 14d = 99}$$

b/ $\frac{25}{2}(2a + 24d) = 300$

$$25a + 300d = 300$$

$$\underline{a + 12d = 12}$$

Sim. eq. $\underline{a = 30}$ $\underline{d = -\frac{3}{2}}$

$$S_{21} = \frac{21}{2}\left(2(30) + 20\left(-\frac{3}{2}\right)\right)$$

$$= \underline{315}$$

c/ $U_{21} = 30 + 20\left(-\frac{3}{2}\right)$

$$= 0$$

The 21st term is zero, all terms after will be negative.

28 $U_n = \sum_{k=0}^{n}(k^2+1) - \sum_{k=0}^{n-1}(k^2+1)$ where n is a positive integer.

(a) Find U_3 and U_8

(b) Solve $U_n = 730$

a/ $U_3 = \sum_{k=0}^{3}(k^2+1) - \sum_{k=0}^{2}(k^2+1)$

$= (1+2+5+10) - (1+2+5)$

$= \underline{\underline{10}}$

$U_8 = \sum_{k=0}^{8}(k^2+1) - \sum_{k=0}^{7}(k^2+1)$

$= 8^2 + 1$

$= \underline{\underline{65}}$

b/ $n^2 + 1 = 730$

$n^2 = 729$

$n = 27$

---

29 Consecutive terms of a sequence are related by $U_{n+1} = (U_n - 3)^2 + 2$

(a) In the case where $U_1 = 2$

(i) Find U_3

(ii) Find U_{20}

(b) State a different value of U_1 that would give the same value for U_{20} found in part (a)(ii)

a i) $U_2 = (2-3)^2 + 2$

$= 3$

$U_3 = (3-3)^2 + 2$

$= 2$

ii/ $U_{20} = 3$ (all even terms will be 3)

b/ 4

30 (a) An arithmetic series is given by $\sum_{r=5}^{20}(3r+4)$

(i) Write down the first term of the series

(ii) Write down the common difference

(iii) Find the number of terms in the series

i/ $3(5)+4 = \underline{\underline{19}}$

ii/ 3

iii/ 16

---

31 An arithmetic series is given by $\sum_{n=11}^{50}(bn+c)$ where b and c are constants

The sum of the series is 470

(a) Show that $122b + 4c = 47$ **(4)**

Given the 21st term is 4 times the 3rd term.

(b) Find the values of b and c. **(4)**

a/ $11b + c + 12b + c + 13b + c + \ldots + 49b + c + 50b + c$

40 terms $\quad S_n = \frac{n}{2}(2a+(n-1)d) \quad d = b$

$\frac{40}{2}\left(2(11b+c) + 39b\right) = 470$

$20(22b + 2c + 39b) = 470$

$2(61b + 2c) = 47$

$122b + 4c = 47$

b/ $31b + c = 4(13b+c)$

$31b + c = 52b + 4c$

$-3c = 21b$

$c = -7b$

$122b + 4(-7b) = 47$

$94b = 47$

$\underline{\underline{b = \frac{1}{2}}}$ $\quad \underline{\underline{c = -\frac{7}{2}}}$

32 The sum to infinity of a geometric series is 108

The first term of the series is less than 40 $u_1 < 40$

The second term of the series is 24 $u_2 = 24$

(a) Find the first term and common ratio of the series.

(b) (i) Show that the nth term of the series, u_n, can be written as $u_n = \frac{2^{n+1}}{3^{n-3}}$

(ii) Hence show that $\log_3 u_n = n(\log_3 2 - 1) + \log_3 2 + 3$

$$\frac{a}{1-r} = 108 \qquad ar = 24$$

$$\boxed{a} = 108 - 108r \qquad \boxed{a = \frac{24}{r}}$$

$$\frac{24}{r} = 108 - 108r$$

$$24 = 108r - 108r^2$$

$$108r^2 - 108r + 24 = 0$$

$$r = \frac{2}{3} \quad \text{or} \quad r = \frac{1}{3} \;\times$$

$$a = 36 \quad \text{or} \quad a = 72 \;\times$$

as a is less than 40, $\underline{\underline{a = 36}}$, $\underline{\underline{r = \frac{2}{3}}}$

b i) $u_n = 36\left(\frac{2}{3}\right)^{n-1}$

$36 = 9 \times 4$
$= 3^2 \times 2^2$

$$= 3^2 \times 2^2 \left(\frac{2^{n-1}}{3^{n-1}}\right)$$

$$= \frac{3^2 \times 2^2 \times 2^{n-1}}{3^{n-1}}$$

$$= \frac{2^2 \times 2^{n-1}}{3^{-2} \times 3^{n-1}}$$

$$= \frac{2^{n+1}}{3^{n-3}}$$

$$\log_3 U_n = \log_3\left(\frac{2^{n+1}}{3^{n-3}}\right)$$

$$\log_3 U_n = \log_3 2^{n+1} - \log_3 3^{n-3}$$

$$\log_3 U_n = (n+1)\log_3 2 - (n-3)$$

$$= n\log_3 2 + \log_3 2 - n + 3$$

$$= n(\log_3 2 - 1) + \log_3 2 + 3$$

---

33 Brad is training to cycle long distances. In Brad's first cycle he travels 25 km. In Brad's second cycle he travels 26 km. Brad believes the distance of his cycles can be modelled by a geometric progression.

(a) Brad sets himself a target of cycling 40 km show that this model predicts Brad will achieve this on his thirteenth cycle.

(b) After six months Brad has travelled a total of 980 km. Use this model to find how many cycles Brad has completed.

a/ $a = 25 \qquad r = \frac{26}{25} = 1.04$

$$U_{11} = 25(1.04)^{10} = 37.0 \text{ km}$$
$$U_{12} = 25(1.04)^{11} = 38.5 \text{ km}$$
$$U_{13} = 25(1.04)^{12} = 40.0 \text{ km}$$

b/ $$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$980 = \frac{25(1.04^n - 1)}{1.04 - 1}$$

$$39.2 = 25(1.04^n - 1)$$

$$1.568 = 1.04^n - 1$$

$$2.568 = 1.04^n$$

$$n = \log_{1.04} 2.568$$
$$= 24.0$$

34 Charlie is training to run a marathon. Each week he goes for a long run. In Charlie's first long run he runs 10 km. He then increases the length of his run by 2 km each week.
A marathon is 42 km.

(a) Find the number of weeks it takes until Charlie runs a full marathon. **(2)**

(b) Find the number of weeks until the **total** length of Charlie's long runs exceeds 200 km. **(4)**

a/

$u_n = a + (n-1)d$

$42 = 10 + (n-1)2$

$42 = 10 + 2n - 2$

$42 = 8 + 2n$

$34 = 2n$

$\underline{\underline{n = 17}}$ (the 17th week, the 17th run)

(16 weeks after the first run)

b/

$\frac{n}{2}(2(10) + (n-1)2) > 200$

$\frac{n}{2}(20 + 2n - 2) > 200$

$\frac{n}{2}(18 + 2n) > 200$

$9n + n^2 > 200$

$n^2 + 9n - 200 > 0$

$n < -19.3 \quad n > 10.3$

×

<u>week 11</u> (10 weeks after the first run)

---

35 A sequence $u_1, u_2, u_3, \ldots$ is defined by $u_n = 20 \times 0.85^n$

Use an algebraic method to find the least possible value for N for which $\sum_{n=1}^{\infty} u_n - \sum_{n=1}^{N} u_n < 1$

$a = 17 \quad r = 0.85$

$S_\infty = \frac{17}{1 - 0.85} = \frac{340}{3}$

37 The first n terms of an arithmetic series are 21 + 34 + 47 + 60 + ... + 177 + 190.

(a) Find the value of n.

(b) Find the sum of these n terms.

$a = 21 \quad d = 13$

$190 = 21 + (n-1)13$

$190 = 21 + 13n - 13$

$190 = 8 + 13n$

$182 = 13n$

$n = 14$

b/ $S_{14} = \frac{14}{2}(21 + 190)$

$= 1477$

---

38 A geometric series is 12 + 9.6 + 7.68 + …

Find the sum to infinity.

$r = \frac{9.6}{12} = 0.8$

$S_\infty = \frac{12}{1 - 0.8}$

$= 60$

$$\frac{340}{3} - \left(\frac{17(1-0.85^N)}{1-0.85}\right) < 1$$

$$\frac{337}{3} < \frac{17(1-0.85^N)}{0.15}$$

$$\frac{337}{340} < 1 - 0.85^N$$

$$0.85^N < \frac{3}{340}$$

$$\log_{0.85}\left(\frac{3}{340}\right) = 29.1$$

$$\underline{N = 30}$$

---

36 An arithmetic series has first term 820 and 5th term 540

(a) Show that the 15th term of the series is negative

(b) The sum of the first n terms is denoted by S. Find the greatest value of S as n varies.

a/ $a = 820$

$$U_5 = a + 4d$$

$$540 = 820 + 4d$$

$$-280 = 4d$$

$$\underline{d = -70}$$

$$U_{15} = 820 + 14(-70)$$

$$\underline{= -160}$$

b/ $U_{14} = 820 + 13(-70)$

$= -90$

$U_{13} = -20$

$U_{12} = 50$

Highest value of S is S_{12}

$$S_{12} = \frac{12}{2}\left(2(820) + 11(-70)\right) = \underline{5220}$$