1 **a** $y = \int (x+2)^3 \ \mathrm{d}x$

$y = \frac{1}{4}(x+2)^4 + c$

b $y = \int 4\cos 2x \ \mathrm{d}x$

$y = 2\sin 2x + c$

c $x = \int 3\mathrm{e}^{2t} + 2 \ \mathrm{d}t$

$x = \frac{3}{2}\mathrm{e}^{2t} + 2t + c$

d $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2-x}$

$y = \int \frac{1}{2-x} \ \mathrm{d}x$

$y = -\ln|2-x| + c$

e $N = \int t\sqrt{t^2+1} \ \mathrm{d}t$

$N = \frac{1}{2}\int 2t(t^2+1)^{\frac{1}{2}} \ \mathrm{d}t$

$N = \frac{1}{2}\times\frac{2}{3}(t^2+1)^{\frac{3}{2}} + c$

$N = \frac{1}{3}(t^2+1)^{\frac{3}{2}} + c$

f $y = \int x\mathrm{e}^x \ \mathrm{d}x$

$u = x, \ \frac{\mathrm{d}u}{\mathrm{d}x} = 1; \ \frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^x, \ v = \mathrm{e}^x$

$y = x\mathrm{e}^x - \int \mathrm{e}^x \ \mathrm{d}x$

$y = x\mathrm{e}^x - \mathrm{e}^x + c$ $\quad [\, y = \mathrm{e}^x(x-1) + c \,]$

2 **a** $y = \int \mathrm{e}^{-x} \ \mathrm{d}x$

$y = -\mathrm{e}^{-x} + c$

$y = 3$ when $x = 0$

$\therefore \ 3 = -1 + c$

$c = 4$

$\therefore \ y = 4 - \mathrm{e}^{-x}$

b $y = \int \tan^3 t \sec^2 t \ \mathrm{d}t$

$y = \frac{1}{4}\tan^4 t + c$

$y = 1$ when $t = \frac{\pi}{3}$

$\therefore \ 1 = \frac{1}{4}(\sqrt{3})^4 + c$

$c = 1 - \frac{9}{4} = -\frac{5}{4}$

$\therefore \ y = \frac{1}{4}\tan^4 t - \frac{5}{4}$ $\quad [\, y = \frac{1}{4}(\tan^4 t - 5) \,]$

c $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{4x}{x^2-3}$

$u = \int \frac{4x}{x^2-3} \ \mathrm{d}x = 2\int \frac{2x}{x^2-3} \ \mathrm{d}x$

$u = 2\ln|x^2-3| + c$

$u = 5$ when $x = 2$

$\therefore \ 5 = 0 + c$

$c = 5$

$\therefore \ u = 2\ln|x^2-3| + 5$

d $y = \int 3\cos^2 x \ \mathrm{d}x$

$y = \frac{3}{2}\int (1 + \cos 2x) \ \mathrm{d}x$

$y = \frac{3}{2}(x + \frac{1}{2}\sin 2x) + c = \frac{3}{4}(2x + \sin 2x) + c$

$y = \pi$ when $x = \frac{\pi}{2}$

$\therefore \ \pi = \frac{3}{4}(\pi + 0) + c$

$c = \frac{\pi}{4}$

$\therefore \ y = \frac{3}{4}(2x + \sin 2x) + \frac{\pi}{4}$

$[\, y = \frac{1}{4}(6x + 3\sin 2x + \pi) \,]$

3 **a** $\frac{x-8}{x^2-x-6} \equiv \frac{A}{x-3} + \frac{B}{x+2}$

$x - 8 \equiv A(x+2) + B(x-3)$

$x = 3 \ \Rightarrow \ -5 = 5A \ \Rightarrow \ A = -1$

$x = -2 \ \Rightarrow \ -10 = -5B \ \Rightarrow \ B = 2$

$\frac{x-8}{x^2-x-6} \equiv \frac{2}{x+2} - \frac{1}{x-3}$

b $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x-8}{x^2-x-6}$

$y = \int \frac{x-8}{x^2-x-6} \ \mathrm{d}x = \int \left(\frac{2}{x+2} - \frac{1}{x-3}\right) \mathrm{d}x$

$y = 2\ln|x+2| - \ln|x-3| + c$

$y = \ln 9$ when $x = 1$

$\therefore \ \ln 9 = 2\ln 3 - \ln 2 + c$

$c = \ln 2 \quad (\ln 9 = \ln 3^2 = 2\ln 3)$

$\therefore \ y = 2\ln|x+2| - \ln|x-3| + \ln 2$

when $x = 2$, $y = 2\ln 4 - 0 + \ln 2 = \ln(4^2 \times 2)$

$= \ln 32$

4 **a** $\int \frac{1}{2y+3}\,dy = \int dx$

$\frac{1}{2}\ln|2y+3| = x + c$

$[\ y = \frac{1}{2}(ke^{2x} - 3)\ ]$

b $\int \text{cosec}^2 2y\,dy = \int dx$

$-\frac{1}{2}\cot 2y = x + c$

$[\ \cot 2y = k - 2x\ ]$

c $\int \frac{1}{y}\,dy = \int x\,dx$

$\ln|y| = \frac{1}{2}x^2 + c$

$[\ y = k e^{\frac{1}{2}x^2}\ ]$

d $\int \frac{1}{y}\,dy = \int \frac{1}{x+1}\,dx$

$\ln|y| = \ln|x+1| + c$

$[\ y = k(x+1)\ ]$

e $\int y\,dy = \int (x^2 - 2)\,dx$

$\frac{1}{2}y^2 = \frac{1}{3}x^3 - 2x + c$

$[\ y^2 = \frac{2}{3}x^3 - 4x + k\ ]$

f $\int \sec^2 y\,dy = \int 2\cos x\,dx$

$\tan y = 2\sin x + c$

g $\int e^{3-y}\,dy = \int x^{-\frac{1}{2}}\,dx$

$-e^{3-y} = 2x^{\frac{1}{2}} + c$

$[\ y = 3 - \ln(k - 2\sqrt{x})\ ]$

h $y\frac{dy}{dx} = x(y^2 + 3)$

$\int \frac{y}{y^2+3}\,dy = \int x\,dx$

$\frac{1}{2}\int \frac{2y}{y^2+3}\,dy = \int x\,dx$

$\frac{1}{2}\ln|y^2+3| = \frac{1}{2}x^2 + c$

$[\ y^2 = k e^{x^2} - 3\ ]$

i $\int \frac{1}{y}\,dy = \int x\sin x\,dx$

$u = x,\ \frac{du}{dx} = 1;\ \frac{dv}{dx} = \sin x,\ v = -\cos x$

$\ln|y| = -x\cos x + \int \cos x\,dx$

$\ln|y| = \sin x - x\cos x + c$

$[\ y = ke^{\sin x - x\cos x}\ ]$

j $\frac{dy}{dx} = \frac{e^{2x}}{e^y}$

$\int e^y\,dy = \int e^{2x}\,dx$

$e^y = \frac{1}{2}e^{2x} + c$

$[\ y = \ln(\frac{1}{2}e^{2x} + c)\ ]$

k $\int \frac{y-3}{y(y-1)}\,dy = \int x\,dx$

$\frac{y-3}{y(y-1)} \equiv \frac{A}{y} + \frac{B}{y-1}$

$y - 3 \equiv A(y-1) + By$

$y = 0 \Rightarrow A = 3,\ y = 1 \Rightarrow B = -2$

$\int (\frac{3}{y} - \frac{2}{y-1})\,dy = \int x\,dx$

$3\ln|y| - 2\ln|y-1| = \frac{1}{2}x^2 + c$

l $\int y^{-2}\,dy = \int \ln x\,dx$

$u = \ln x,\ \frac{du}{dx} = \frac{1}{x};\ \frac{dv}{dx} = 1,\ v = x$

$-y^{-1} = x\ln x - \int dx$

$-y^{-1} = x\ln x - x + c$

$[\ y = \frac{1}{x - x\ln x + k}\ ]$

5 **a** $\int 2y \; dy = \int x \; dx$

$y^2 = \frac{1}{2}x^2 + c$

$y = 3$ when $x = 4$

$\therefore \; 9 = 8 + c$

$c = 1$

$\therefore \; y^2 = \frac{1}{2}x^2 + 1$

b $\int (y+1)^{-3} \; dy = \int dx$

$-\frac{1}{2}(y+1)^{-2} = x + c$

$(y+1)^{-2} = k - 2x$

$y = 0$ when $x = 2$

$\therefore \; 1 = k - 4$

$k = 5$

$\therefore \; (y+1)^{-2} = 5 - 2x$

$[\ (y+1)^2 = \frac{1}{5-2x}\ ]$

c $\int \frac{1}{y} \; dy = \int \cot^2 x \; dx$

$\int \frac{1}{y} \; dy = \int (\operatorname{cosec}^2 x - 1) \; dx$

$\ln|y| = -\cot x - x + c$

$y = 1$ when $x = \frac{\pi}{2}$

$\therefore \; 0 = 0 - \frac{\pi}{2} + c$

$c = \frac{\pi}{2}$

$\therefore \; \ln|y| = \frac{\pi}{2} - \cot x - x$

d $\int \frac{1}{y+2} \; dy = \int \frac{1}{x-1} \; dx$

$\ln|y+2| = \ln|x-1| + c$

$y = 6$ when $x = 3$

$\therefore \; \ln 8 = \ln 2 + c$

$c = \ln 4$

$\therefore \; \ln|y+2| = \ln|x-1| + \ln 4$

$[\ y + 2 = 4(x-1) \Rightarrow y = 4x - 6\ ]$

e $\int \cot y \; dy = \int x^2 \; dx$

$\int \frac{\cos y}{\sin y} \; dy = \int x^2 \; dx$

$\ln|\sin y| = \frac{1}{3}x^3 + c$

$y = \frac{\pi}{6}$ when $x = 0$

$\therefore \; \ln \frac{1}{2} = 0 + c$

$c = -\ln 2$

$\therefore \; \ln|\sin y| = \frac{1}{3}x^3 - \ln 2$

$[\ 2\sin y = e^{\frac{1}{3}x^3}\ ]$

f $\frac{dy}{dx} = \frac{\sqrt{y}}{\sqrt{x+3}}$

$\int y^{-\frac{1}{2}} \; dy = \int (x+3)^{-\frac{1}{2}} \; dx$

$2y^{\frac{1}{2}} = 2(x+3)^{\frac{1}{2}} + c$

$\sqrt{y} = \sqrt{x+3} + k$

$y = 16$ when $x = 1$

$\therefore \; 4 = 2 + k$

$k = 2$

$\therefore \; \sqrt{y} = \sqrt{x+3} + 2$

$[\ y = (\sqrt{x+3} + 2)^2\ ]$

g $\int \sin y \; dy = \int xe^{-x} \; dx$

$u = x, \; \frac{du}{dx} = 1; \; \frac{dv}{dx} = e^{-x}, \; v = -e^{-x}$

$-\cos y = -xe^{-x} + \int e^{-x} \; dx$

$-\cos y = -xe^{-x} - e^{-x} + c$

$\cos y = (x+1)e^{-x} + k$

$y = \pi$ when $x = -1$

$\therefore \; -1 = 0 + k$

$k = -1$

$\therefore \; \cos y = (x+1)e^{-x} - 1$

h $\int \frac{\sin y}{1+\cos y} \; dy = \int \frac{1}{2}x^{-2} \; dx$

$-\int \frac{-\sin y}{1+\cos y} \; dy = \int \frac{1}{2}x^{-2} \; dx$

$-\ln|1 + \cos y| = -\frac{1}{2}x^{-1} + c$

$\ln|1 + \cos y| = \frac{1}{2}x^{-1} + k$

$y = \frac{\pi}{3}$ when $x = 1$

$\therefore \; \ln \frac{3}{2} = \frac{1}{2} + k$

$k = \ln \frac{3}{2} - \frac{1}{2}$

$\therefore \; \ln|1 + \cos y| = \frac{1}{2}x^{-1} + \ln \frac{3}{2} - \frac{1}{2}$

$[\ (1 + \cos y)^2 = \frac{9}{4}e^{\frac{1-x}{x}}\ ]$

1 **a** $\frac{(x+4)}{(1+x)(2-x)} \equiv \frac{A}{1+x} + \frac{B}{2-x}$

$x + 4 \equiv A(2-x) + B(1+x)$

$x = -1 \Rightarrow A = 1,\ x = 2 \Rightarrow B = 2$

$\therefore\ \frac{(x+4)}{(1+x)(2-x)} \equiv \frac{1}{1+x} + \frac{2}{2-x}$

b $\int \frac{1}{y}\ dy = \int (\frac{1}{1+x} + \frac{2}{2-x})\ dx$

$\ln|y| = \ln|1+x| - 2\ln|2-x| + c$

$y = 2$ when $x = 3$

$\therefore\ \ln 2 = \ln 4 - 0 + c$

$c = -\ln 2$

$\therefore\ \ln|y| = \ln|1+x| - 2\ln|2-x| - \ln 2$

$[\ y = \frac{1+x}{2(2-x)^2}\ ]$

2 $\frac{dy}{dx} = e^y \times e^x \cos x$

$\int e^{-y}\ dy = \int e^x \cos x\ dx$

$u = e^x,\ \frac{du}{dx} = e^x;\ \frac{dv}{dx} = \cos x,\ v = \sin x$

$I = \int e^x \cos x\ dx = e^x \sin x - \int e^x \sin x\ dx$

$u = e^x,\ \frac{du}{dx} = e^x;\ \frac{dv}{dx} = \sin x,\ v = -\cos x$

$I = e^x \sin x - (-e^x \cos x + \int e^x \cos x\ dx)$

$2I = e^x \sin x + e^x \cos x + C$

$-e^{-y} = \frac{1}{2}e^x(\sin x + \cos x) + c$

$e^{-y} = k - \frac{1}{2}e^x(\sin x + \cos x)$

$y = 0$ when $x = 0$

$\therefore\ 1 = k - \frac{1}{2}$

$k = \frac{3}{2}$

$\therefore\ e^{-y} = \frac{3}{2} - \frac{1}{2}e^x(\sin x + \cos x)$

$[\ 2e^{-y} = 3 - e^x(\sin x + \cos x)\ ]$

3 $\frac{dy}{dx} = \frac{k}{x}$

$\frac{dy}{dx} = \frac{5}{3}$ when $x = 3$

$\therefore\ \frac{5}{3} = \frac{k}{3},\ k = 5$

$y = \int \frac{5}{x}\ dx$

$y = 5\ln|x| + c$

$y = 4$ when $x = 3$

$\therefore\ 4 = 5\ln 3 + c$

$c = 4 - 5\ln 3$

$\therefore\ y = 5\ln|x| + 4 - 5\ln 3$

$y = 5\ln|\frac{x}{3}| + 4$

4 **a** $\frac{dN}{dt} = kN$

b $\int \frac{1}{N}\ dN = \int k\ dt$

$\ln|N| = kt + c$

$N = e^{kt+c} = e^c \times e^{kt}$

$N = Ae^{kt}$

c $t = 0, N = 40\ \therefore\ A = 40$

$t = 5, N = 60\ \therefore\ 60 = 40e^{5k}$

$\therefore\ k = \frac{1}{5}\ln\frac{3}{2} = 0.0811$ (3sf)

d $t = 12\ \therefore\ N = 40e^{0.08109 \times 12}$

$= 106$ (3sf)

5 **a** let side length be l

$A = 6l^2 \;\therefore\; l = \sqrt{\frac{A}{6}}$

$V = l^3 = (\sqrt{\frac{A}{6}})^3 = 6^{-\frac{3}{2}} A^{\frac{3}{2}}$

$\therefore\; \frac{dV}{dA} = \frac{3}{2} \times 6^{-\frac{3}{2}} \times A^{\frac{1}{2}} = k\sqrt{A}$

b $\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$

$\frac{dV}{dt} = cA$

$\therefore\; cA = k\sqrt{A} \times \frac{dA}{dt}$

$\frac{dA}{dt} = d\sqrt{A}$

$A = 100,\; \frac{dA}{dt} = 5 \quad \therefore\; d = \frac{1}{2}$

$\frac{dA}{dt} = \frac{1}{2}\sqrt{A}$

$\int 2A^{-\frac{1}{2}}\, dA = \int dt$

$4A^{\frac{1}{2}} = t + C$

$t = 10,\; A = 100 \quad \therefore\; C = 30$

$A^{\frac{1}{2}} = \frac{1}{4}(t + 30)$

$A = \frac{1}{16}(t + 30)^2$

6 **a** $-\frac{dm}{dt} = km$

$\int \frac{1}{m}\, dm = \int -k\, dt$

$\ln|m| = -kt + c$

$m = e^{-kt + c} = e^c \times e^{-kt}$

$m = Ae^{-kt}$

$t = 0,\; m = 24 \;\therefore\; A = 24$

$m = 24e^{-kt}$

b $t = 20,\; m = 22.6 \;\therefore\; 22.6 = 24e^{-20k}$

$\therefore\; k = -\frac{1}{20}\ln\frac{22.6}{24} = 0.00301$ (3sf)

c $\frac{dm}{dt} = -km = -0.003005 \times 22.6$

$= -0.0679$ (3sf)

$\therefore$ decreasing at 0.0679 grams per day

d $m = 12 \;\therefore\; 12 = 24e^{-0.003005t}$

$t = -\frac{1}{0.003005}\ln\frac{1}{2}$

$= 231$ days (nearest day)

7 **a** $-\frac{dP}{dt} = k\sqrt{P}$

$\int P^{-\frac{1}{2}}\, dP = \int -k\, dt$

$2P^{\frac{1}{2}} = -kt + c$

$\sqrt{P} = \frac{1}{2}c - \frac{1}{2}kt = a - bt$

$\therefore\; P = (a - bt)^2$

b $t = 0,\; P = 400 \quad \therefore\; \sqrt{400} = a - 0$

$a = 20$

c $t = 30,\; P = 100 \;\therefore\; \sqrt{100} = 20 - 30b$

$b = \frac{1}{3}$

$\therefore\; P = (20 - \frac{1}{3}t)^2$

$t = 50 \;\therefore\; P = (20 - \frac{50}{3})^2$

$= 11\frac{1}{9}$

8 **a** $-\frac{dV}{dt} = aV$ where a is a positive constant

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

if angle at vertex $= 2\theta,\; \tan\theta = \frac{r}{h}$

$\therefore\; r = bh$ where b is a positive constant

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi b^2 h^3 \quad \therefore\; \frac{dV}{dh} = \pi b^2 h^2$

$\therefore\; -\pi b^2 h^2 \times \frac{dh}{dt} = a \times \frac{1}{3}\pi b^2 h^3$

$\frac{dh}{dt} = -kh$ where k is a positive constant

b $\int \frac{1}{h}\, dh = \int -k\, dt$

$\ln|h| = -kt + c$

$h = e^{-kt + c} = e^c \times e^{-kt} = Ae^{-kt}$

$t = 0,\; h = 12 \quad \therefore\; A = 12$

$t = 20,\; h = 10 \quad \therefore\; 10 = 12e^{-20k}$

$\therefore\; k = -\frac{1}{20}\ln\frac{5}{6}$

$\therefore\; h = 12e^{-kt},\; k = 0.00912$ (3sf)

c $6 = 12e^{-0.009116t}$

$t = -\frac{1}{0.009116}\ln\frac{1}{2} = 76.0$ (3sf)

9 **a** $\frac{1}{(1+x)(1-x)} \equiv \frac{A}{1+x} + \frac{B}{1-x}$

$1 \equiv A(1-x) + B(1+x)$

$x = -1 \Rightarrow A = \frac{1}{2},\ x = 1 \Rightarrow B = \frac{1}{2}$

$\frac{1}{(1+x)(1-x)} \equiv \frac{1}{2(1+x)} + \frac{1}{2(1-x)}$

b $t = 0,\ m = 0,\ \frac{dm}{dt} = 0.5$

$\therefore\ 0.5 = k \times 1$

$k = 0.5$

c $\int \frac{1}{(1+m)(1-m)}\ dm = \int \frac{1}{2}e^{-t}\ dt$

$\int \left(\frac{\frac{1}{2}}{1+m} + \frac{\frac{1}{2}}{1-m}\right)\ dm = \int \frac{1}{2}e^{-t}\ dt$

$\frac{1}{2}\ln|1+m| - \frac{1}{2}\ln|1-m| = -\frac{1}{2}e^{-t} + c$

$\ln|1+m| - \ln|1-m| = C - e^{-t}$

$t = 0,\ m = 0\ \therefore\ 0 - 0 = C - 1$

$C = 1$

$\ln|1+m| - \ln|1-m| = 1 - e^{-t}$

for $0 \le m < 1,\ 1 + m > 0$ and $1 - m > 0$

$\therefore\ \ln(1+m) - \ln(1-m) = 1 - e^{-t}$

$\ln\left(\frac{1+m}{1-m}\right) = 1 - e^{-t}$

d $m = 0.1\ \therefore\ \ln\frac{1.1}{0.9} = 1 - e^{-t}$

$t = -\ln(1 - \ln\frac{11}{9}) = 0.2240$ hrs

$= 13.4$ minutes

e $t \to \infty,\ \ln\left(\frac{1+m}{1-m}\right) \to 1$

$\therefore$ limiting value of m is given by

$\frac{1+m}{1-m} = e$

$1 + m = e(1-m)$

$m(1+e) = e - 1$

$m = \frac{e-1}{1+e} = 0.4621$

$\therefore$ max. produced ≈ 462 g