C4 Differentiation

1a) $y^2 - 3y = x^3 + 8$

$2y\frac{dy}{dx} - 3\frac{dy}{dx} = 3x^2$

$\frac{dy}{dx}(2y - 3) = 3x^2$

$\frac{dy}{dx} = \frac{3x^2}{2y-3}$

b) when $y = 3$

$(3)^2 - 3(3) = x^3 + 8$

$0 = x^3 + 8$

$-8 = x^3$

$x = -2$

$\frac{dy}{dx} = \frac{3(-2)^2}{2(3)-3}$

$= \underline{4}$

2a) $V = \frac{1}{3}\pi r^2 h$ $\quad r = \frac{2}{3}h$ [similar shapes]

$= \frac{1}{3}\pi\left(\frac{2}{3}h\right)^2 h$

$= \frac{1}{3}\pi\left(\frac{4}{9}h^2\right)h$

$= \frac{4}{27}\pi h^3$

$= \frac{4\pi h^3}{27}$

b) $\frac{dV}{dt} = 8$ $\quad \frac{dV}{dh} = \frac{12\pi}{27}h^2$

$\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$

$= 8 \times \frac{27}{12\pi h^2}$

$= 8 \times \frac{27}{12\pi(12)^2} = \underline{\frac{1}{8}\pi}$

3 $ye^{-2x} = 2x + y^2$

$u = y \qquad v = e^{-2x}$

$\frac{du}{dx} = \frac{dy}{dx} \quad \frac{dv}{dx} = -2e^{-2x}$

$e^{-2x}\frac{dy}{dx} + -2ye^{-2x} = 2 + 2y\frac{dy}{dx}$

$e^{-2x}\frac{dy}{dx} - 2y\frac{dy}{dx} = 2 + 2ye^{-2x}$

$\frac{dy}{dx}(e^{-2x} - 2y) = 2 + 2ye^{-2x}$

$\frac{dy}{dx} = \frac{2 + 2ye^{-2x}}{e^{-2x} - 2y}$

b/ at $(0,1)$

$\frac{dy}{dx} = \frac{2 + 2(1)e^{-2(0)}}{e^{-2(0)} - 2(1)}$

$= -4$

$m = \frac{1}{4}$ (perpendicular)

$y = \frac{1}{4}x + c$

$y = \frac{1}{4}x + 1$

$4y = x + 4$

$x - 4y + 4 = 0$

4/ $\frac{dA}{dt} = 0.032$

$A = \pi x^2$

$\frac{dA}{dx} = 2\pi x$

$\frac{dx}{dt} = \frac{dx}{dA} \times \frac{dA}{dt}$

$= \frac{1}{2\pi x} \times 0.032$

when $x = 2$

$$\frac{dx}{dt} = \frac{1}{2\pi(2)} \times 0.032$$

$$= \frac{1}{125\pi} = 0.0025\dagger \text{ cm/s (3sf)}$$

$$= 2.55 \times 10^{-3} \text{ cm/s} \quad 3sf$$

b/ $V = \pi(x)^2(5x)$

$$= 5\pi x^3$$

$$\frac{dV}{dx} = 15\pi x^2$$

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$

$$= 15\pi x^2 \times \frac{1}{2\pi x} \times 0.032$$

$$= 15\pi(2)^2 \times \frac{1}{2\pi(2)} \times 0.032$$

$$= 4.74 \text{ cm}^3/\text{s} \quad (3sf)$$

$$= 0.48 \text{ cm}^3/\text{s}$$

5a) $3x^2 - y^2 + xy = 4$

$u = x \quad v = y$

$\frac{du}{dx} = 1 \quad \frac{dy}{dx} = \frac{dy}{dx}$

$$6x - 2y\frac{dy}{dx} + x\frac{dy}{dx} + y = 0$$

when $\frac{dy}{dx} = 8/3$

$$6x - 2y\left(\frac{8}{3}\right) + \frac{8}{3}x + y = 0$$

$$\frac{26}{3}x - \frac{13}{3}y = 0$$

$$26x - 13y = 0$$

$$2x - y = 0$$

$$y - 2x = 0$$

b/ $y = 2x$

$$3x^2 - (2x)^2 + x(2x) = 4$$

$$3x^2 - 4x^2 + 2x^2 = 4$$

$x^2 = 4$

$x = \pm 2$

$y = 2(2)$ $\quad$ $y = 2(-2)$

$= 4$ $\quad$ $= -4$

$(2, 4)$ and $(-2, -4)$

6) $x^3 - 4y^2 = 12xy \qquad x = -8$

$(-8)^3 - 4y^2 = 12(-8)y$

$-512 - 4y^2 = -96y$

$0 = 4y^2 - 96y + 512$

$0 = y^2 - 24y + 128$

$0 = (y - 16)(y - 8)$

$y = 16 \qquad y = 8$

$\underline{(-8, 8)} \qquad \underline{(-8, 16)}$

b/ $x^3 - 4y^2 = 12xy$

$3x^2 - 8y\frac{dy}{dx} = 12x\frac{dy}{dx} + 12y$

$3x^2 - 12y = 12x\frac{dy}{dx} + 8y\frac{dy}{dx}$

$3x^2 - 12y = \frac{dy}{dx}(12x + 8y)$

$\frac{dy}{dx} = \frac{3x^2 - 12y}{12x + 8y}$

at $(-8, 8)$

$\frac{dy}{dx} = \frac{3(-8)^2 - 12(8)}{12(-8) + 8(8)}$

$= \underline{-3}$

at $(-8, 16)$

$\frac{dy}{dx} = \frac{3(-8) - 12(16)}{12(-8) + 8(16)}$

$= \underline{0}$

7 $\sin x + \cos y = 0.5$

$\cos x + -\sin(y)\frac{dy}{dx} = 0$

$\cos x - \frac{dy}{dx}\sin y = 0$

$\cos x = \frac{dy}{dx}\sin y$

$\frac{dy}{dx} = \frac{\cos x}{\sin y}$

b/ $\frac{dy}{dx} = \cos x\left(\frac{1}{\sin y}\right)$

$0 = \cos x\left(\frac{1}{\sin y}\right)$

$\cos x = 0 \qquad \frac{1}{\sin y} = 0$

$x = \frac{1}{2}\pi, -\frac{1}{2}\pi$

when $x = \frac{1}{2}\pi$

$\sin(\frac{1}{2}\pi) + \cos y = 0.5$

$\cos y = 0.5 - \sin(\frac{1}{2}\pi)$

$y = \frac{2}{3}\pi, -\frac{2}{3}\pi$

when $x = -\frac{1}{2}\pi$

$\cos y = \frac{3}{2}$

NO SOLUTION.

$\left(\frac{1}{2}\pi, \frac{2}{3}\pi\right)$ and $\left(\frac{1}{2}\pi, \frac{-2}{3}\pi\right)$

8a/ $y = 2^x$

$\ln y = \ln 2^x$

$\ln y = x \ln 2$

$\frac{1}{y}\frac{dy}{dx} = \ln 2$

$\frac{dy}{dx} = y \ln 2$

$\frac{dy}{dx} = 2^x \ln 2$

b/ $y = 2^{(x^2)}$

$\frac{dy}{dx} = 2^{(x^2)} \ln 2 \times 2x$

when $x = 2$

$\frac{dy}{dx} = 2^4 \ln 2 \times 4$

$= 64 \ln 2$

9/ $3x^2 - 2y^2 + 2x - 3y + 5 = 0$

$6x - 4y\frac{dy}{dx} + 2 - 3\frac{dy}{dx} = 0$

$6x + 2 = 4y\frac{dy}{dx} + 3\frac{dy}{dx}$

$x \quad y$

$(0, 1)$

$6x + 2 = \frac{dy}{dx}(4y + 3)$

$2 = \frac{dy}{dx}(7)$

$\frac{dy}{dx} = \frac{2}{7}$

$m = -\frac{7}{2}$

$y = -7/2\, x + 1$

$2y = -7x + 2$

$2y + 7x - 2 = 0$

10/ $3x^2 + 4y^2 - 2x + 6xy - 5 = 0$

$$6x + 8y\frac{dy}{dx} - 2 + 6x\frac{dy}{dx} + 6y = 0$$

$x \quad y$

$(1, -2)$ $\quad 6(1) + 8(-2)\frac{dy}{dx} - 2 + 6(1)\frac{dy}{dx} + 6(-2) = 0$

$$6 - 16\frac{dy}{dx} - 2 + 6\frac{dy}{dx} - 12 = 0$$

$$-10\frac{dy}{dx} = 8$$

$$\frac{dy}{dx} = -4/5$$

$$y = -4/5\,x + c$$

$$-2 = -4/5 + c$$

$$-10 = -4 + 5c$$

$$-6 = 5c$$

$$c = -6/5$$

$$y = -4/5\,x - 6/5$$

$$5y = -4x - 6$$

$$\underline{4x + 5y + 6 = 0}$$