C3 Differentiation

1a) $u = x^2 \qquad v = \cos 3x$

$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = -3\sin(3x)$

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$= -3x^2 \sin(3x) + 2x\cos(3x)$

b) $u = \ln(x^2+1) \qquad v = (x^2+1)$

$\frac{du}{dx} = \frac{2x}{(x^2+1)} \qquad \frac{dv}{dx} = 2x$

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{2x - 2x\ln(x^2+1)}{(x^2+1)^2}$

ii) $y = (4x+1)^{1/2}$

$\frac{dy}{dx} = 2(4x+1)^{-1/2}$

when $x = 2 \qquad \frac{dy}{dx} = 2/3$

$y = 3$

$y = 2/3\,x + c \qquad (2, 3)$

$3 = 2/3(2) + c$

$3 = 4/3 + c$

$c = 5/3$

$\therefore\ y = 2/3\,x + 5/3$

$3y = 2x + 5$

$\underline{0 = 2x - 3y + 5}$

2a) $y = x^2 (5x-1)^{1/2}$

$u = x^2 \qquad v = (5x-1)^{1/2}$

$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = \frac{5}{2}(5x-1)^{-1/2}$

$\frac{dy}{dx} = \frac{5}{2}x^2(5x-1)^{-1/2} + 2x(5x-1)^{1/2}$

when $x=2$

$\frac{dy}{dx} = \frac{5}{2}(2)^2(5(2)-1)^{-1/2} + 2(2)(5(2)-1)^{1/2}$

$= 46/3$

b) $\frac{\sin 2x}{x^2}$

$u = \sin 2x \qquad v = x^2$

$\frac{du}{dx} = 2\cos 2x \qquad \frac{dv}{dx} = 2x$

$\frac{dy}{dx} = \frac{2x^2 \cos 2x - 2x \sin 2x}{x^4}$

$= \frac{2x\cos 2x - 2\sin 2x}{x^3}$

3ai) $u = e^{3x} \qquad v = \sin x + 2\cos x$

$\frac{du}{dx} = 3e^{3x} \qquad \frac{dv}{dx} = \cos x - 2\sin x$

$\frac{dy}{dx} = e^{3x}(\cos x - 2\sin x) + 3e^{3x}(\sin x + 2\cos x)$

$= e^{3x}(\cos x - 2\sin x + 3\sin x + 6\cos x)$

$= e^{3x}(\cancel{4}\ 7\cos x + \sin x)$

ii) $u = x^3 \qquad v = \ln(5x+2)$

$$\frac{du}{dx} = 3x^2 \qquad \frac{dv}{dx} = \frac{5}{5x+2}$$

$$\frac{5x^3}{5x+2} + 3x^2 \ln(5x+2)$$

b) $$y = \frac{3x^2 + 6x - 7}{(x+1)^2}$$

$$u = 3x^2 + 6x - 7 \qquad v = (x+1)^2$$

$$\frac{du}{dx} = 6x + 6 \qquad \frac{du}{dx} = 2(x+1)$$

$$\frac{dy}{dx} = \frac{(6x+6)(x+1)^2 - 2(x+1)(3x^2+6x-7)}{(x+1)^4}$$

$$= \frac{(6x+6)(x+1) - 2(3x^2+6x-7)}{(x+1)^3}$$

$$= \frac{6x^2 + 6x + 6x + 6 - 6x^2 - 12x + 14}{(x+1)^3}$$

$$= \frac{20}{(x+1)^3}$$

c) $$\frac{dy}{dx} = 20(x+1)^{-3}$$

$$\frac{d^2y}{dx^2} = -60(x+1)^{-4}$$

$$-60(x+1)^{-4} = -\frac{15}{4}$$

$$\frac{-60}{(x+1)^4} = \frac{-15}{4}$$

$$(x+1)^4 = 16$$

$$x+1 = \pm 2$$

$$x = -1 \pm 2$$

$$\underline{x = 1} \text{ or } \underline{x = -3}$$

4a) $y = e^{2x} \tan x$

$u = e^{2x}$ $\quad v = \tan x$

$\frac{du}{dx} = 2e^{2x}$ $\quad \frac{dv}{dx} = \sec^2 x$

$\frac{dy}{dx} = e^{2x} \sec^2 x + 2e^{2x} \tan x$

turning points are where $\frac{dy}{dx} = 0$

$e^{2x} \sec^2 x + 2e^{2x} \tan x = 0$

$e^{2x}((\sec^2 x) + 2 \tan x) = 0$

$e^{2x}((\tan^2 x + 1) + 2\tan x) = 0$

$e^{2x}(\tan^2 x + 2\tan x + 1) = 0$

$e^{2x}(\tan x + 1)(\tan x + 1) = 0$

$\underline{\tan x = -1}$ $\quad \underline{\tan x = -1}$

b) when $x = 0$

$\frac{dy}{dx} = e^{2(0)} \sec^2(0) + 2e^{2(0)} \tan(0)$

$= 1$

when $x = 0$, $y = 0$

$\underline{\underline{y = x}}$

5a) $y = x^2 e^x$

$u = x^2$ $\quad v = e^x$

$\frac{du}{dx} = 2x$ $\quad \frac{dv}{dx} = e^x$

$\frac{dy}{dx} = x^2 e^x + 2x e^x$

b) turning points where $\frac{dy}{dx} = 0$

$x^2 e^x + 2x e^x = 0$

$e^x(x^2 + 2x) = 0$

$(e^x)(x)(x + 2) = 0$

$x = 0$ $\quad x = -2$

$(0, 0)$ $\quad (-2, 4e^{-2})$

c/ $\frac{dy}{dx} = x^2e^x + 2xe^x$

$u = x^2 \quad v = e^x \qquad u = 2x \quad v = e^x$

$\frac{d^2y}{dx^2} \quad \frac{du}{dx} = 2x \quad \frac{dv}{dx} = e^x \quad \frac{du}{dx} = 2 \quad \frac{dv}{dx} = e^x$

$\frac{d^2y}{dx^2} = x^2e^x + 2xe^x + 2xe^x + 2e^x$

$= x^2e^x + 4xe^x + 2e^x$

d/ $x = 0 \quad (0)^2e^0 + 4(0)e^0 + 2e^0 = 2$

+ve ∴ minimum

$x = -2 \quad (-2)^2e^{-2} + 4(-2)e^{-2} + 2e^{-2} = -2e^{-2}$

-ve ∴ maximum

6. $x = 2\sin y$

$\sqrt{2} = 2\sin \pi/4$

$\sqrt{2} = 2\frac{\sqrt{2}}{2}$

$\underline{\sqrt{2} = \sqrt{2}}$ shown.

b/ $\frac{dx}{dy} = 2 \cancel{\sin} \cos y$

when $y = \pi/4$

$\frac{dx}{dy} = 2\cos(\pi/4)$

$= 2\frac{\sqrt{2}}{2}$

$= \sqrt{2}$

$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{2}}$

c/ $m = -\sqrt{2}$

$y = -\sqrt{2}\, x + c \qquad (\underset{x}{\sqrt{2}}, \underset{y}{\pi/4})$

$\pi/4 = -\sqrt{2}(\sqrt{2}) + c$

$\pi/4 = -2 + c$

$c = \pi/4 + 2$

$\underline{y = -\sqrt{2}\, x + \pi/4 + 2}$

7i/ $y = \dfrac{x}{9 + x^2}$

$u = x \qquad v = 9 + x^2$

$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 2x$

$\frac{dy}{dx} = \dfrac{9 + x^2 - 2x^2}{(9 + x^2)^2} = \dfrac{9 - x^2}{(9 + x^2)^2}$

turning points where $\frac{dy}{dx} = 0$

$\dfrac{9 + x^2 - 2x^2}{(9 + x^2)^2} = 0$

$9 + x^2 - 2x^2 = 0$

$9 = x^2$

$x = \pm 3$

$y = \pm 1/6$

ii/ $y = (1 + e^{2x})^{3/2}$

$\frac{dy}{dx} = \frac{3}{2} 2e^{2x}(1 + e^{2x})^{1/2}$

$= 3e^{2x}(1 + e^{2x})^{1/2}$

when $x = \frac{1}{2} \ln 3$

$\underline{\frac{dy}{dx} = 18}$

8 $y = e^{3x} + \ln 2x$

$\frac{dy}{dx} = 3e^{3x} + \frac{1}{x}$

b) $y = (5 + x^2)^{3/2}$

$\frac{dy}{dx} = \frac{3}{2} \cdot 2x(5 + x^2)^{1/2}$

$= 3x(5 + x^2)^{1/2}$

9 $y = \ln\left(\frac{1}{3}x\right)$

$\frac{dy}{dx} = \frac{1}{x}$ when $x = 3$, $y = 0$, $m = 1/3$

perp. $m = -3$

$y = -3x + c$

$0 = -3(3) + c$

$c = 9$

$\underline{\underline{y = -3x + 9}}$

10a/ i/ $u = x^2$ $v = e^{3x+2}$

$\frac{du}{dx} = 2x$ $\frac{dv}{dx} = 3e^{3x+2}$

$\frac{dy}{dx} = 3x^2e^{3x+2} + 2xe^{3x+2}$

ii/ $\frac{\cos(2x^3)}{3x}$

$u = \cos(2x^3)$ $v = 3x$

$\frac{du}{dx} = -6x^2\sin(2x^3)$ $\frac{dv}{dx} = 3$

$\frac{dy}{dx} = \frac{-18x^3\sin(2x^3) - 3\cos(2x^3)}{(3x)^2}$

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b) $x = 4\sin(2y+6)$

$\frac{dx}{dy} = 8\cos(2y+6)$

$\frac{dy}{dx} = \frac{1}{8\cos(2y+6)}$

$\sin^2 x + \cos^2 x = 1$

$\cos^2 x = 1 - \sin^2 x$

$\cos x = \sqrt{1-\sin^2 x}$

$\therefore \frac{dy}{dx} = \frac{1}{8\sqrt{1-\sin^2(2y+6)}}$

$= \frac{1}{8\sqrt{1-\left(\frac{x}{4}\right)^2}}$

11a) $y = 3\sin^2 x + \sec 2x$

$\frac{dy}{dx} = 6\sin x\cos x + 2\sec 2x\tan 2x$

ii) $y = (x+\ln 2x)^3$

$\frac{dy}{dx} = 3(x+\ln 2x)^2\left(1+\frac{1}{x}\right)$

b) $y = \frac{5x^2-10x+9}{(x-1)^2}$

$u = 5x^2-10x+9$ $\quad v = (x-1)^2$

$\frac{du}{dx} = 10x-10$ $\quad \frac{dv}{dx} = 2(x-1)$

$\frac{dy}{dx} = \frac{(10x-10)(x-1)^2 - 2(x-1)(5x^2-10x+9)}{(x-1)^4}$

$= \frac{(10x-10)(x-1) - 2(5x^2-10x+9)}{(x-1)^3}$

$$\frac{10x^2 - 10x - 10x + 10 - 10x^2 + 20x - 18}{(x-1)^3}$$

$$\frac{-8}{(x-1)^3}$$

12 $f(x) = 1 - \frac{2}{(x+4)} + \frac{x-8}{(x-2)(x+4)}$

$$= \frac{(x-2)(x+4)}{(x-2)(x+4)} - \frac{2(x-2)}{(x-2)(x+4)} + \frac{x-8}{(x-2)(x+4)}$$

$$= \frac{x^2 + 4x - 2x - 8 - 2x + 4 + x - 8}{(x-2)(x+4)}$$

$$= \frac{x^2 + x - 12}{(x-2)(x+4)}$$

$$= \frac{(x+4)(x-3)}{(x-2)(x+4)}$$

$$= \frac{x-3}{x-2}$$

b)

$$\frac{e^x - 3}{e^x - 2}$$

$u = e^x - 3 \qquad v = e^x - 2$

$\frac{du}{dx} = e^x \qquad \frac{dv}{dx} = e^x$

$$\frac{dy}{dx} = \frac{e^x(e^x - 2) - e^x(e^x - 3)}{x(e^x - 2)^2}$$

$$= \frac{e^{2x} - 2e^x - e^{2x} + 3e^x}{(e^x - 2)^2}$$

$$= \frac{e^x}{(e^x - 2)^2}$$

e) $$\frac{e^x}{(e^x-2)^2} = 1$$

$$e^x = (e^x-2)^2$$

$$e^x = e^{2x} - 4e^x + 4$$

$$0 = e^{2x} - 5e^x + 4$$

$$0 = (e^x-4)(e^x-1)$$

$$e^x = 4 \qquad e^x = 1$$

$$\underline{x = \ln 4} \qquad \underline{x = 0}$$

13 $$\frac{2x+2}{x^2-2x-3} - \frac{x+1}{x-3}$$

$$\frac{2x+2}{(x-3)(x+1)} - \frac{(x+1)^2}{(x-3)(x+1)}$$

$$\frac{2x+2-(x^2+2x+1)}{(x-3)(x+1)}$$

$$\frac{2x+2-x^2-2x-1}{(x-3)(x+1)}$$

$$\frac{1-x^2}{(x-3)(x+1)}$$

$$\frac{(\cancel{1+x})(1-x)}{(x-3)(\cancel{x+1})}$$

$$\frac{1-x}{x-3}$$

b) $u = 1-x \qquad v = x-3$

$$\frac{du}{dx} = -1 \qquad \frac{dv}{dx} = 1$$

$$\frac{-1(x-3)-(1-x)}{(x-3)^2}$$

$$\frac{-x+3-1+x}{(x-3)^2}$$

$$\frac{2}{(x-3)^2}$$

14a/ $\dfrac{2x+3}{x+2} - \dfrac{9+2x}{2x^2+3x-2}$

$\dfrac{2x+3}{x+2} - \dfrac{9+2x}{(2x-1)(x+2)}$

$\dfrac{(2x+3)(2x-1)}{(x+2)(2x-1)} - \dfrac{(9+2x)}{(x+2)(2x-1)}$

$\dfrac{(2x+3)(2x-1) - (9+2x)}{(x+2)(2x-1)}$

$\dfrac{4x^2-2x+6x-3-9-2x}{(x+2)(2x-1)}$

$\dfrac{4x^2+2x-12}{(x+2)(2x-1)}$

$\dfrac{2(2x-3)(x+2)}{(x+2)(2x-1)}$

$\dfrac{2(2x-3)}{(2x-1)}$

$\dfrac{4x-6}{2x-1}$

b/ $u = 4x-6 \qquad v = 2x-1$

$\dfrac{du}{dx} = 4 \qquad \dfrac{dv}{dx} = 2$

$\dfrac{dy}{dx} = \dfrac{4(2x-1) - 2(4x-6)}{(2x-1)^2}$

$= \dfrac{8x-4-8x+12}{(2x-1)^2}$

$= \dfrac{8}{(2x-1)^2}$