C4 Parametric Equations

1a) $x = t^3 - 8t$ $\quad\quad$ $y = t^2$

$t=-1$ $\quad x = (-1)^3 - 8(-1)$ $\quad\quad$ $y = (-1)^2$

$= 7$ $\quad\quad$ $= 1$

$\underline{\underline{(7,1)}}$

b) $\dfrac{dx}{dt} = 3t^2 - 8$ $\quad\quad$ $\dfrac{dy}{dt} = 2t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{2t}{3t^2 - 8}$$

$t = -1$ $\quad \dfrac{dy}{dx} = \dfrac{2(-1)}{3(-1)^2 - 8}$

$$= \frac{-2}{-5} = \frac{2}{5}$$

$y = \frac{2}{5}x + c$ $\quad\quad (7,1)$

$$1 = \frac{2}{5}(7) + c$$

$$1 = \frac{14}{5} + c$$

$$c = -\frac{9}{5}$$

$$y = \frac{2}{5}x - \frac{9}{5}$$

$$5y = 2x - 9$$

$$\underline{\underline{2x - 5y - 9 = 0}}$$

c) $2(t^3 - 8t) - 5t^2 - 9 = 0$

$2t^3 - 16t - 5t^2 - 9 = 0$

$2t^3 - 5t^2 - 16t - 9 = 0$

$f(1) = -28$

$f(-1) = 0 \qquad (t+1)$ is a factor

$$\begin{array}{r|l} & 2t^2 - 7t - 9 \\ \hline t+1 & 2t^3 - 5t^2 - 16t - 9 \\ & 2t^3 + 2t^2 \\ & \quad -7t^2 - 16t \\ & \quad -7t^2 - 7t \\ & \qquad\quad -9t - 9 \\ & \qquad\quad -9t - 9 \\ \hline & \qquad\qquad 0 \end{array}$$

$(t+1)(2t^2 - 7t - 9)$

$(t+1)(2t-9)(t+1)$

$t = -1 \qquad t = 9/2$

$x = \left(\frac{9}{2}\right)^3 - 8\left(\frac{9}{2}\right) \qquad y = \left(\frac{9}{2}\right)^2$

$= \frac{441}{8} \qquad = \frac{81}{4}$

$\left(\frac{441}{8}, \frac{81}{4}\right)$

2a) $\frac{dx}{dt} = -4 \sin 2t \qquad \frac{dy}{dt} = 6 \cos t$

$\frac{dy}{dx} = \frac{6 \cos t}{-4 \sin 2t}$

$t = \frac{\pi}{3} \qquad = \frac{6 \cos \pi/3}{-4 \sin\left(\frac{2\pi}{3}\right)}$

$= -\frac{\sqrt{3}}{2}$

b) $x = 2 \cos 2t$

$= 2(\cos^2 t - \sin^2 t)$

$= 2(\cos^2 t - (1 - \cos^2 t))$

$= 2(2\cos^2 t - 1)$

$= 4 \cos^2 t - 2$

$$\frac{x+2}{4} = \cos^2 t \qquad \left(\frac{y}{6}\right)^2 = \sin^2 t$$

$$\cos^2 t + \sin^2 t = 1$$

$$\frac{x+2}{4} + \left(\frac{y}{6}\right)^2 = 1$$

$$\frac{x+2}{4} + \frac{y^2}{36} = 1$$

$$9\,\cancel{36}\frac{(x+2)}{\cancel{4}} + y^2 = 36$$

$$y^2 = 36 - 9x - 18$$

$$y^2 = 18 - 9x$$

$$y = \sqrt{18-9x} \qquad k=2$$

c/ $0 \leq f(x) \leq 6$

3a/ $\int \sin^2\theta \, d\theta$

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$2\sin^2\theta = 1 - \cos 2\theta$$

$$\sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta)$$

$$\int \tfrac{1}{2} - \tfrac{1}{2}\cos 2\theta \, d\theta$$

$$= \tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta + c$$

b/ $\pi \int y^2 \frac{dx}{d\theta} \, d\theta$

$$x = \tan\theta$$

$$\frac{dx}{d\theta} = \sec^2\theta$$

$$\pi \int (2\sin 2\theta)^2 \cancel{t} \sec^2\theta \, d\theta$$

$$\pi \int 4\sin^2 2\theta \cdot \sec^2\theta \, d\theta$$

$$4\pi \int (2\sin\theta\cos\theta)^2 \frac{1}{\cos^2\theta} \, d\theta$$

$$4\pi \int 4\sin^2\theta\cos^2\theta \times \frac{1}{\cos^2\theta}\, d\theta$$

$$16\pi \int \sin^2\theta\, d\theta$$

c/ $$16\pi\left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta + c\right]_0^{\frac{1}{6}\pi}$$

when $x = \frac{1}{\sqrt{3}}$

$\frac{1}{\sqrt{3}} = \tan\theta$

$\theta = \frac{1}{6}\pi$

$$16\pi\left[\left(\tfrac{1}{2}\cdot\tfrac{1}{6}\pi - \tfrac{1}{4}\sin\left(\tfrac{1}{3}\pi\right)\right) - \left(\tfrac{1}{2}(0) - \tfrac{1}{4}\sin 0\right)\right]$$

$$16\pi\left(\tfrac{1}{12}\pi - \tfrac{\sqrt{3}}{8}\right)$$

$$\tfrac{16}{12}\pi^2 - 2\sqrt{3}\,\pi$$

$$\underline{\underline{\tfrac{4}{3}\pi^2 - 2\pi\sqrt{3}}}$$

4/a $(4, 2\sqrt{3})$

$x = 8\cos t$

$4 = 8\cos t$

$\frac{1}{2} = \cos t$

$t = \underline{\underline{\frac{1}{3}\pi}}$

b/ $\frac{dx}{dt} = -8\sin t$ $\qquad \frac{dy}{dt} = 8\cos 2t$

$$\frac{dy}{dx} = \frac{8\cos 2t}{-8\sin t}$$

$$= -\frac{\cos 2t}{\sin t}$$

at P $\quad \frac{dy}{dx} = \frac{\sqrt{3}}{3}$ $\qquad \therefore\ m = \frac{-3}{\sqrt{3}} = \underline{\underline{-\sqrt{3}}}$

~~$(4, 2\sqrt{3})$~~ $\cancel{y = \frac{\sqrt{3}}{3}x + c}$

$\cancel{2\sqrt{3} = \frac{\sqrt{3}}{3}(4) + c}$

$y = -\sqrt{3}x + c$

$2\sqrt{3} = -4\sqrt{3} + c$

$6\sqrt{3} = c$

$$\underline{\underline{y = -\sqrt{3}x + 6\sqrt{3}}}$$

$$\int y \frac{dx}{dt} \, dt$$

$$\int_{\frac{1}{2}\pi}^{\frac{1}{3}\pi} (4 \sin 2t)(-8 \sin t) \, dt$$

$$\int_{\frac{1}{2}\pi}^{\frac{1}{3}\pi} 4(2 \sin t \cos t)(-8 \sin t) \, dt$$

$$\int_{\frac{1}{2}\pi}^{\frac{1}{3}\pi} -64 \sin^2 t \cos t \, dt.$$

$$\int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi} 64 \sin^2 t \cos t \, dt$$

d/ Let $y = (\sin t)^3$

$$\frac{dy}{dx} = 3(\sin t)^2 \cos t$$

$\therefore$

$$\left[\frac{64}{3} (\sin^3 t)\right]_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}$$

$$\left[\frac{64}{3}\left(\sin \tfrac{1}{2}\pi\right)^3\right] - \left[\frac{64}{3}\left(\sin\left(\tfrac{1}{3}\pi\right)\right)^3\right]$$

$$= \frac{64 - 24\sqrt{3}}{3}$$

$$= \frac{64}{3} - 8\sqrt{3}$$

5a/ $\int_{\ln 2}^{\ln 4} y \, dx$

$\ln(2) = \ln(t+2)$

$t = 0$

$\ln 4 = \ln(t+2)$

$t = 2$

$$\int_0^2 y \frac{dx}{dt} \, dt$$

$$\int_0^2 \frac{1}{t+1} \cdot \frac{1}{t+2} \, dt$$

$\frac{dx}{dt} = \frac{1}{t+2}$

$$\int_0^2 \frac{1}{(t+1)(t+2)} \, dt$$

b/ $\int_0^2 \frac{1}{(t+1)(t+2)}\,dt$

$$\frac{1}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$$

$$1 = A(t+2) + B(t+1)$$

Let $\cancel{t} = -1$

$1 = A$

Let $t = -2$ $\quad 1 = -B$

$B = -1$

$$\int_0^2 \frac{1}{(t+1)} - \frac{1}{(t+2)}\,dt$$

$$\Big[\ln(t+1) - \ln(t+2)\Big]_0^2$$

$$\Big[\ln(3) - \ln(4)\Big] - \Big[\ln(1) - \ln(2)\Big]$$

$$= \ln\left(\tfrac{3}{4}\right) - \ln\left(\tfrac{1}{2}\right)$$

$$= \ln\left(\frac{3/4}{1/2}\right) = \underline{\ln\left(\tfrac{3}{2}\right)}$$

c/ $x = \ln(t+2)$

$e^x = t+2$

$e^x - 2 = t$

$$y = \frac{1}{e^x - 2 + 1}$$

$$y = \frac{1}{e^x - 1}$$

d/ $x > \ln(-1+2)$

$\underline{x > 0}$

6) a) $x = \tan^2 t$ $\quad y = \sin t$

$\frac{dx}{dt} = 2\tan t \sec^2 t$ $\quad \frac{dy}{dt} = \cos t$

$$\frac{dy}{dx} = \frac{\cos t}{2\tan t \sec^2 t} = \cancel{\frac{\cos^2 t \sin t}{2}} \quad \frac{\cos^4 t}{2\sin t}$$

b) where $t = \pi/4$

$$\frac{dy}{dx} = \cancel{\frac{\left(\cos\left(\frac{\pi}{4}\right)\right)^2\left(\sin\left(\frac{\pi}{4}\right)\right)}{2}} \quad \frac{\cos\left(\frac{\pi}{4}\right)}{2\tan\left(\frac{\pi}{4}\right)\sec^2\left(\frac{\pi}{4}\right)}$$

$$= \frac{\sqrt{2}}{8}$$

$$y = \frac{\sqrt{2}}{8}x + c$$

$$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{8}\left(\tan\frac{\pi}{4}\right)^2 + c$$

$$c = \frac{3\sqrt{2}}{8}$$

$$y = \frac{\sqrt{2}}{8}x + \frac{3\sqrt{2}}{8}$$

c) $y^2 = \sin^2 t$ $\quad x = \tan^2 t$

$y^2 = 1 - \cos^2 t$ $\quad x = \sec^2 t - 1$

$y^2 = 1 - \frac{1}{x+1}$ $\quad x = \frac{1}{\cos^2 t} - 1$

$$x + 1 = \frac{1}{\cos^2 t}$$

$$\cos^2 t = \frac{1}{x+1}$$

$$y^2 = 1 - \frac{1}{x+1}$$

7a) $x = 7\cos t - \cos 7t$ $\quad y = 7\sin t - \sin 7t$

$\frac{dx}{\cancel{dy}\,dt} = -7\sin t + 7\sin 7t$ $\quad \frac{dy}{dt} = 7\cos t - 7\cos 7t$

$$\frac{dy}{dx} = \frac{7\cos t - 7\cos 7t}{-7\sin t + 7\sin 7t}$$

b) where $t = \pi/6$

$$\frac{dy}{dx} = \frac{7\cos(\pi/6) - 7\cos(\frac{7\pi}{6})}{-7\sin(\pi/6) + 7\sin(7\pi/6)}$$

$$= -\sqrt{3}$$

$\therefore m = \frac{1}{\sqrt{3}}$ $\quad y = \frac{1}{\sqrt{3}}x + c$

$$7\sin(\tfrac{\pi}{6}) - \sin(\tfrac{7\pi}{6}) = \tfrac{1}{\sqrt{3}}\left(7\cos(\tfrac{\pi}{6}) - \cos(\tfrac{7\pi}{6})\right) + c$$

$$4 = \frac{1}{\sqrt{3}}(4\sqrt{3}) + c$$

$$4 = 4 + c$$

$$c = 0$$

$$\underline{\underline{y = \frac{1}{\sqrt{3}}x}}$$

8a) $x = \overset{\sin t}{\cancel{\sin t}}$ $\quad y = \sin(t + \pi/6)$

$\frac{dx}{dt} = \cos t$ $\quad \frac{dy}{dt} = \cos(t + \pi/6)$

$$\frac{dy}{dx} = \frac{\cos(t + \pi/6)}{\cos t}$$

at $t = \pi/6$ $\quad \frac{dy}{dx} = \frac{\cos(\pi/3)}{\cos(\pi/6)}$

$$= \frac{\sqrt{3}}{3}$$

$$y = \frac{\sqrt{3}}{3}x + c$$

$y = \sin\left(\frac{\pi}{3}\right)$ $\qquad x = \sin\left(\pi/6\right)$

$= \frac{\sqrt{3}}{2}$ $\qquad x = 1/2$

$$\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{3} \cdot \frac{1}{2} + c$$

$$\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{6} + c$$

$$c = \frac{\sqrt{3}}{3}$$

$$y = \frac{\sqrt{3}}{3}x + \frac{\sqrt{3}}{3}$$

b/ $y = \sin\left(t + \pi/6\right)$

$= \sin t \cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{6}\right)\cos t$

$= \frac{\sqrt{3}}{2} \cancel{\sin t} \sin t + \frac{1}{2}\cos t$

$= \frac{\sqrt{3}}{2} x + \frac{1}{2}\cos t$

$= \underline{\underline{\frac{\sqrt{3}}{2} x + \frac{1}{2}\sqrt{1 - x^2}}}$

$x = \sin t$

$x^2 = \sin^2 t$

$x^2 = 1 - \cos^2 t$

$\cos^2 t = 1 - x^2$

$\cos t = \sqrt{1 - x^2}$

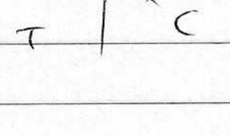
9a crosses x when $y = 0$

$0 = 1 - 2\cos t$

$2\cos t = 1$

$\cos t = 1/2$

$t = \underline{\underline{\frac{1}{3}\pi, \frac{5}{3}\pi}}$



b/ $\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} y \frac{dx}{dt} \, dt$ $\qquad x = t - 2\sin t$

$\frac{dx}{dt} = 1 - 2\cos t$

$\int_{\frac{\pi}{3}}^{\frac{5\pi}{3}} (1-2\cos t)(1-2\cos t) \, dt$

$\int_{\pi/3}^{5\pi/3} (1-2\cos t)^2 \, dt.$

c/ $\int_{\pi/3}^{\frac{5\pi}{3}} (1-2\cos t)(1-2\cos t) \, dt$

$\int 1 - 4\cos t + 4\cos^2 t \quad dt$

$\int 1 - 4\cos t + 2\cos 2t + 2 \quad dt$

$\int_{\pi/3}^{5\pi/3} 3 - 4\cos t + 2\cos 2t \, dt$

$\cos 2t = 2\cos^2 t - 1$
$\cos 2t + 1 = 2\cos^2 t$
$2\cos 2t + 2 = 4\cos^2 t$

$\left[3t - 4\sin(t) + \sin(2t) + c\right]_{\pi/3}^{5\pi/3}$

$\left[3\left(\frac{5\pi}{3}\right) - 4\sin\left(\frac{5\pi}{3}\right) + \sin\left(\frac{10\pi}{3}\right)\right] - \left[3\left(\frac{\pi}{3}\right) - 4\sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{2\pi}{3}\right)\right]$

$\left(5\pi + 2\sqrt{3} - \frac{\sqrt{3}}{2}\right) - \left(\pi - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right)$

$4\pi + 4\sqrt{3} - \sqrt{3}$

$\underline{4\pi + 3\sqrt{3}}$